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BIFURCATION DIAGRAMS FOR HAMILTONIAN LINEAR TYPE CENTERS OF LINEAR PLUS CUBIC HOMOGENEOUS POLYNOMIAL VECTOR FIELDS

ILKER E. COLAK, JAUME LLIBRE, AND CLAUDIA VALLS

Abstract. As a natural continuation of the work done in [7] we provide the bifurcation diagrams for the global phase portraits in the Poincaré disk of all the Hamiltonian linear type centers of linear plus cubic homogeneous planar polynomial vector fields.

1. Introduction and statement of the main results

Limit cycles and, being closely related, the center–focus problem have been among the main subjects that recently attracted a lot of attention in the qualitative theory of real planar differential systems. The center–focus problem refers to determining whether a singular point is a center of a focus. The definition of center was first introduced by Poincaré in [18]. He defined a center as a singular point of a vector field on the real plane which has a neighborhood that consists solely of periodic orbits and the singular point itself.

Analytic differential systems having a center at the origin are grouped in three categories. If after an affine change of variables and a rescaling of the time variable the differential system can be written in the form

$$\dot{x} = -y + P(x, y), \quad \dot{y} = x + Q(x, y),$$

then it is called a *linear type center*; if it can be written in the form

$$\dot{x} = y + P(x, y), \quad \dot{y} = Q(x, y),$$

then it is called a *nilpotent center*; and finally if it can be written in the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

then it is called a *degenerate center*. Here $P(x, y)$ and $Q(x, y)$ are real analytic functions without constant and linear terms, defined in a neighborhood of the origin. For the characterization of linear type centers Poincaré [19] and Lyapunov [15] provide an algorithm, we also refer to Chazy [5] and Moussu [17]. On the other hand an algorithm for the characterization of nilpotent and some class of degenerate centers are given by Chavarriga *et al.* [4], Cima and Llibre [6], Giacomini *et al.* [11], and Giné and Llibre [12].

The classification of centers of quadratic polynomial differential systems started with the works of Dulac [9], Kapteyn [13, 14] and Bautin [1]. In [22] Vulpe provides all the global phase portraits of quadratic polynomial

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differential systems having a center. Schlomiuk [20] and Żołądek [25] provided the bifurcation diagrams of all quadratic differential systems having a center.

Considering the classification of the centers of polynomial differential systems with degrees higher than two there are many but partial results. For linear type centers of cubic polynomial differential systems having linear terms with homogeneous nonlinearities of degree three were characterized by Malkin [16], and by Vulpe and Sibirski [23]. We provide all the global phase portraits of Hamiltonian linear type and nilpotent centers of linear plus cubic homogeneous polynomial vector fields in [7] and [8], repectively. In addition we refer to Rousseau and Schlomiuk [21], and Żołądek [26, 27] for some interesting results in some subclasses of cubic systems. Systems with higher degrees homogeneous nonlinearities the linear type centers are not fully characterized, but see Chavarriga and Giné [2, 3] for some of the main results. In any case there is still a long way to fully characterize and classify the centers of all polynomial differential systems of degree three.

In this work we provide the bifurcation diagrams for the global phase portraits in the Poincaré disk of all the Hamiltonian linear type centers of linear plus cubic homogeneous planar polynomial vector fields. We say that two vector fields on the Poincaré disk are *topologically equivalent* if there exists a homeomorphism from one onto the other which sends orbits to orbits preserving or reversing the direction of the flow. In [8] the global phase portraits of all Hamiltonian planar polynomial vector fields with only linear and cubic homogeneous terms having a linear center at the origin are given by the following theorem:

Theorem 1. *Any Hamiltonian linear type planar polynomial vector field with linear plus cubic homogeneous terms has a linear type center at the origin if and only if, after a linear change of variables and a rescaling of its independent variable, it can be written as one of the following six classes:*

(I) $\dot{x} = ax + by,\ \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3$

(II) $\dot{x} = ax + by - x^3,\ \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3x^2y,$

(III) $\dot{x} = ax + by - 3x^2y + y^3,\ \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3xy^2,$

(IV) $\dot{x} = ax + by - 3x^2y - y^3,\ \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3xy^2,$

(V) $\dot{x} = ax + by - 3\mu x^2y + y^3,\ \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3 + 3\mu xy^2,$

(VI) $\dot{x} = ax + by - 3\mu x^2y - y^3,\ \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3 + 3\mu xy^2,$

where $a, b, \beta, \mu \in \mathbb{R}$ with $b \neq 0$ and $\beta > 0$. Moreover, the global phase portraits of these six families of systems are topologically equivalent to the following of Figure 1:

(a) 1.1 *or* 1.2 *for systems (I);*
(b) 1.3 *for systems (II);*
(c) 1.4, 1.5 *or* 1.6 *for systems (III);*
(d) 1.1, 1.2, 1.7, 1.8 *or* 1.9 *for systems (IV);*

(e) 1.3, 1.10, 1.11 *or* 1.12 *for systems* (V)*;*
(f) 1.13–1.23 *for systems* (VI)*.*

We remark that using the change of variables $(u, v) = (x/\sqrt{\beta}, y/\sqrt{\beta})$, the time rescale $d\tau = \beta dt$, and redefining parameters $\bar{a} = a/\beta$ and $\bar{b} = b/\beta$, we can assume $\beta = 1$ in the families of systems (I)–(VI). We also note that in the families (III)–(VI) the cases with $a < 0$ are obtained from those with $a > 0$ simply by making the change $(t, x) \mapsto (-t, -x)$. Therefore we will assume $a \geq 0$ for these systems. We state our main result as follows:

Theorem 2. *The global phase portraits of Hamiltonian planar polynomial vector fields with linear plus cubic homogeneous terms having a linear type center at the origin are topologically equivalent to the following ones of Figure 1 using the notation of Theorem 1:*

(a) *For systems* (I) *the phase portrait is*
   1.1 *when* $b < 0$*;*
   1.2 *when* $b > 0$*.*
(b) *For systems* (II) *the unique phase portrait is* 1.3*.*
(c) *For systems* (III) *the phase portrait is*
   1.4 *when* $b < 0$*;*
   1.5 *when* $b > 0$ *and* $a = 0$*;*
   1.6 *when* $b > 0$ *and* $a > 0$*.*
   The corresponding bifurcation diagram is shown in Figure 2.
(d) *For systems* (IV) *the phase portrait is*
   1.1 *when* $b < 0$*;*
   1.2 *when* $b > 0$*,* $D = 0$ *and* $a = 0$*, or when* $b > 0$ *and* $D > 0$*;*
   1.7 *when* $b > 0$*,* $D < 0$ *and* $a = 0$*;*
   1.8 *when* $b > 0$*,* $D < 0$ *and* $a > 0$*;*
   1.9 *when* $b > 0$*,* $D = 0$ *and* $a > 0$*.*
   See (6) *for the definition of* D*. The corresponding bifurcation diagram is shown in Figure 3.*
(e) *For systems* (V) *we can assume* $b > 0$*, and the phase portrait is*
   1.3 *when* $\mu \leq 0$*, or when* $\mu > 0$ *and* $D_4 < 0$*, or when* $\mu > 0$*,* $D_4 = 0$ *and* $a = 0$*;*
   1.10 *when* $\mu > 0$*,* $D_4 > 0$ *and* $a = 0$*;*
   1.11 *when* $\mu > 0$*,* $D_4 > 0$ *and* $a > 0$*;*
   1.12 *when* $\mu > 0$*,* $D_4 = 0$ *and* $a > 0$*.*
   See (16) *for the definition of* D_4*. The corresponding bifurcation diagram for the case* $\mu > 0$ *is shown in Figure 4.*
(f) *For systems* (VI) *we can assume* $b > 0$ *whenever* $\mu < -1/3$*, and the phase portrait is*
   1.13 *when* $\mu < -1/3$ *and* $b \neq \sqrt{1 + a^2}$*;*
   1.14 *when* $\mu < -1/3$ *and* $b = \sqrt{1 + a^2}$*;*
   1.15 *when* $\mu = -1/3$ *and* $b < 0$*;*
   1.16 *when* $\mu = -1/3$*,* $b > 0$ *and* $b \neq \sqrt{1 + a^2}$*;*
   1.17 *when* $\mu = -1/3$ *and* $b = \sqrt{1 + a^2}$*;*
   1.18 *when* $\mu > -1/3$ *and* $b < 0$*;*
   1.19 *when* $\mu > -1/3$*,* $b > 0$*,* $D_4 < 0$*, or when* $\mu > -1/3$*,* $b > 0$*,* $D_4 = D_3 = 0$ *and either* $a \neq 0$ *or* $\mu \neq 1/3$ *or* $b \neq 1$*;*

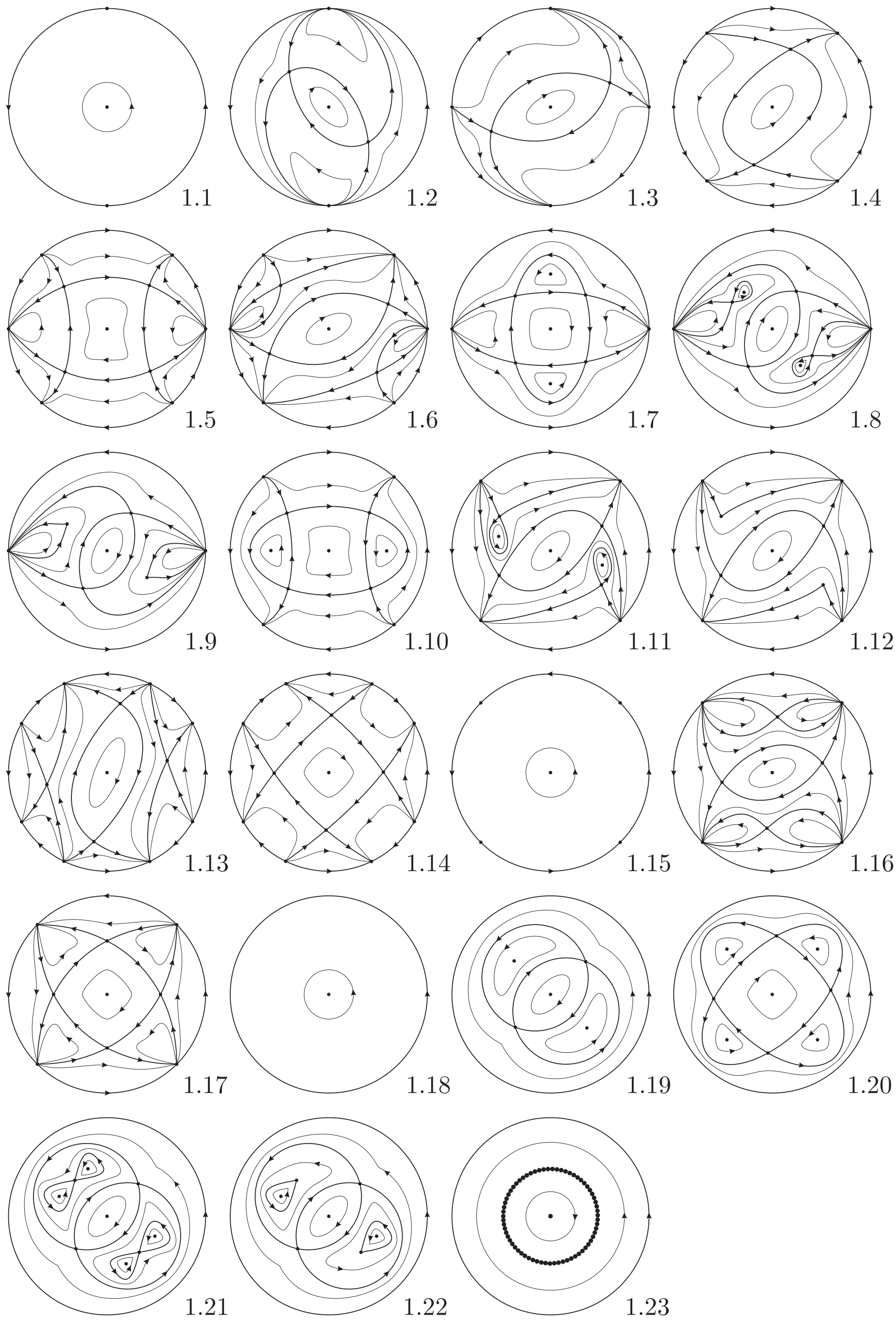

FIGURE 1. Global phase portraits of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a linear type center at the origin. The separatrices are in bold.

1.20 *when* $1/3-2a/(3\sqrt{1+a^2}) > \mu > -1/3$*,* $D_4 > 0$ *and* $b = \sqrt{1+a^2}$*, or when* $\mu > 1/3$*,* $b > 0$*,* $D_4 > 0$ *and* $a = 0$*;*
1.21 *when* $\mu > -1/3$*,* $b > 0$*,* $D_4 > 0$ *and* $b \neq \sqrt{1+a^2}$*, or when* $\mu > 1/3 + 2a/(3\sqrt{1+a^2})$*,* $b = \sqrt{1+a^2}$*,* $D_4 > 0$ *and* $a \neq 0$*;*
1.22 *when* $\mu > -1/3$*,* $b > 0$*,* $D_4 = 0$ *and* $D_3 \neq 0$*;*
1.23 *when* $a = 0$*,* $\mu = 1/3$ *and* $b = 1$*.*

See (30) *and* (43) *for the definitions of* D_4 *and* D_3*, respectively. The corresponding bifurcation diagrams are shown in Figures 5–9.*

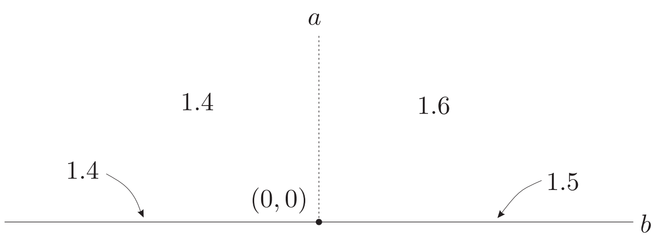

FIGURE 2. Bifurcation diagram for systems (III).

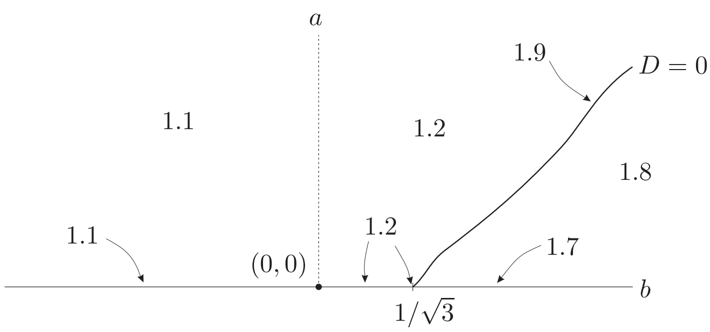

FIGURE 3. Bifurcation diagram for systems (IV).

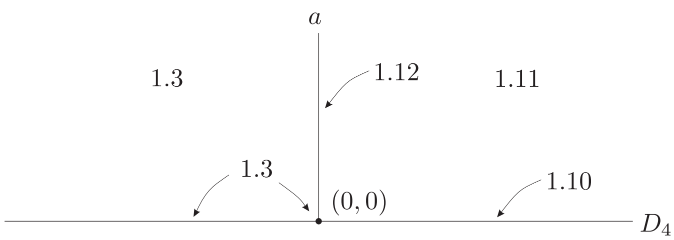

FIGURE 4. Bifurcation diagram for systems (V) with $\mu > 0$.

We note that the bifurcation diagrams for the centers of Theorem 2 in the particular case when they are reversible were also given in [10].

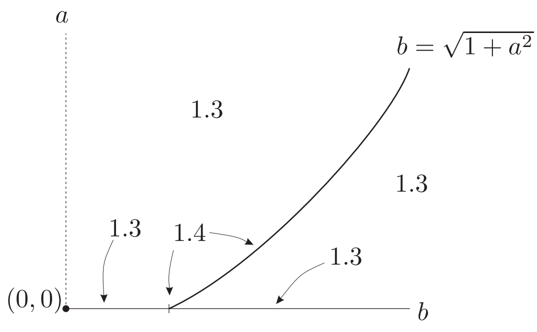

FIGURE 5. Bifurcation diagram for systems (VI) with $\mu < -1/3$ and $b > 0$.

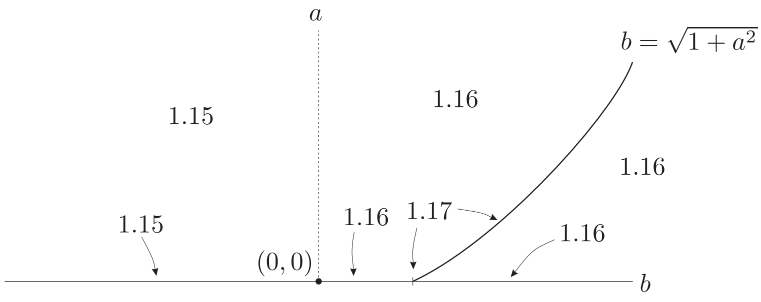

FIGURE 6. Bifurcation diagram for systems (VI) with $\mu = -1/3$.

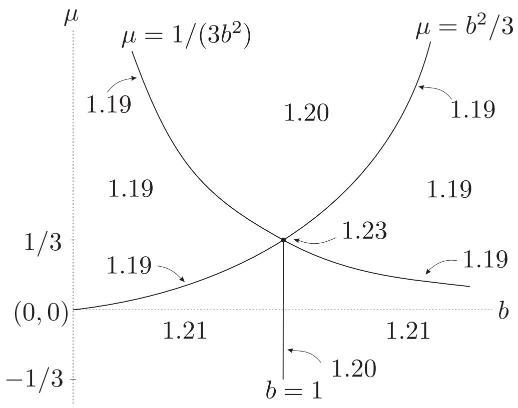

FIGURE 7. Bifurcation diagram for systems (VI) with $\mu > -1/3$ and $a = 0$.

Statement (a) of Theorem 2 follows directly from the result obtained in [7]. Furthermore systems (II), up to topological equivalence, have a unique global phase portrait. However [7] provides very little information to obtain the full bifurcation diagrams for the families (III)–(VI). Therefore in this work we focus on these families.

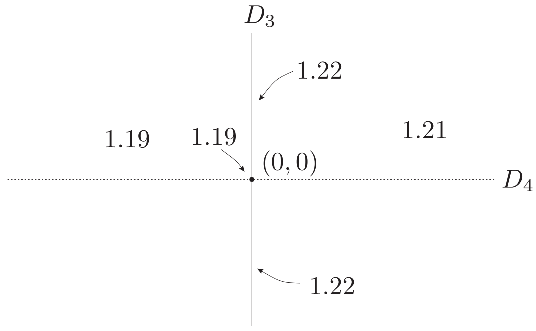

FIGURE 8. Bifurcation diagram for systems (VI) with $\mu > -1/3$, $a > 0$ and $b \neq \sqrt{1+a^2}$.

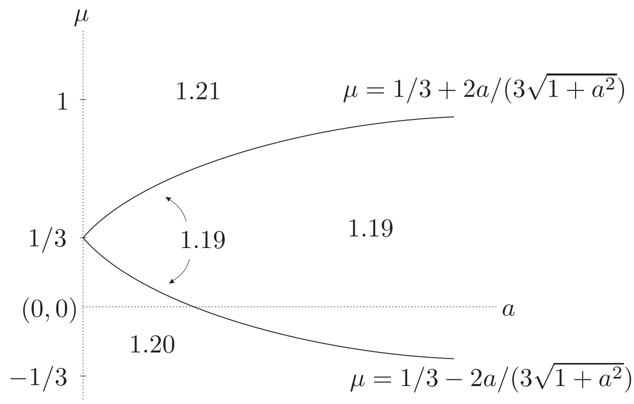

FIGURE 9. Bifurcation diagram for systems (VI) with $\mu > -1/3$, $a > 0$, $b = \sqrt{1+a^2}$ and $D_4 > 0$.

2. Bifurcation diagram for systems (III)

Systems (III)

$$\dot{x} = ax + by - 3x^2y + y^3, \tag{1a}$$

$$\dot{y} = -\frac{a^2+1}{b}x - ay + 3xy^2, \tag{1b}$$

have the Hamiltonian

$$H_3(x,y) = \frac{y^4}{4} - \frac{3}{2}x^2y^2 + \frac{a^2+1}{2b}x^2 + \frac{b}{2}y^2 + axy.$$

In [7] it is shown that each phase portrait of systems (III) is topologically equivalent to the phase portrait 1.4 of Figure 1 when $b < 0$, and to either 1.5 or 1.6 when $b > 0$. Thus we only need to determine the bifurcation values of parameter a in the case $b > 0$ leading to either the phase portrait 1.5 or the phase portrait 1.6. So we assume $b > 0$.

We define the *energy levels* of a Hamiltonian differential system as the level sets of its Hamiltonian. We make the following remark.

Remark 3. There can be at most two finite saddles at a fixed energy level in the phase portrait 1.6. Indeed if in the phase portrait 1.6 all four saddles were at the same energy level, then a straight line through the origin passing close enough to the saddles that are not on the boundary of the period annulus of the center at the origin would intersect the separatrices of these saddles six times. Since these separatrices are at the same energy level this clearly cannot happen as H_3 is quartic, and $H_3(x, cx) - h$ can have at most four roots for any $h \in \mathbb{R}$.

As a result of Remark 3 we see that the phase portraits 1.5 and 1.6 have four and two finite saddles at a fixed energy level, respectively. We will use this observation to distinguish the two phase portraits.

The number of singular points at the same energy level is equal to the number of solutions $\mathcal{N}$ of the system of equations $\dot{x} = \dot{y} = H_3 - h = 0$ for some $h \in \mathbb{R}$. We note that $h > 0$ at any singular point besides the origin because

$$H_3 - \frac{y\dot{x} - x\dot{y}}{4} = \frac{x^2 + (ax + by)^2}{4b} > 0.$$

To find $\mathcal{N}$ we compute the Gröbner basis of the three polynomials (1a), (1b) and $H_3 - h$, and obtain a set of 23 polynomials. Due to the size of these polynomials we will only mention in this paper the ones which are enough for our purpose. We remark that since $b > 0$ systems (V) don't have any finite singular points on the coordinate axes other than the origin, so we will assume $xy \neq 0$ in our calculations.

There are two polynomials in the Gröbner basis which do not contain the variable x, and they are quadratic in y of the form $my^2 + n$, where m and n are functions of the parameters a and b. The coefficient of y^2 in these polynomials are $6hp_1$ and $3hp_2$, where

$$p_1 = 1 - 90h + 1728h^2 + 5832b^2h^2 + 13824h^3,$$
$$p_2 = 11 + 3a^2 - 18b^2 - 336h - 972b^2h - 2304h^2.$$

We claim that these coefficients cannot vanish simultaneously. In fact if we calculate the resultant of p_1 and p_2 with respect to h we obtain

$$4a^6 + a^4(12 - 45b^2) + 3a^2(4 - 3b^2 + 36b^4) + 4(1 + 3b^2)^3, \tag{2}$$

up to a positive constant. If we consider (2) as a polynomial in a^2 we see that its the discriminant with respect to a^2 is

$$-(1 + 3b^2)^3(16 + 39b^2 + 72b^4)^2 < 0.$$

Hence it has a unique real root. In addition it has at least one negative root due to Descartes' rule of signs because $4 - 3b^2 + 36b^4 > 0$. Then the only real root of (2), when considered as a polynomial in a^2, is negative and consequently it cannot be zero for real a. Therefore p_1 and p_2 cannot be zero at the same time. Since $h > 0$, our claim is proved. Consequently the singular points which are at the same energy level must be on two vertical lines on the real plane.

There is another polynomial in the Gröbner basis which is linear in the variable x, and the coefficient of x is $27a(1+a^2)$. So if $a \neq 0$ we have $\mathcal{N} = 2$.

If $a = 0$ we can simply calculate the finite singular points besides the origin of systems (III) which are

$$\left(\pm\frac{1}{3}\sqrt{\frac{1+3b^2}{b}}, \pm\sqrt{\frac{1}{3b}}\right).$$

Since the Hamiltonian H_3 is even these four points are at the same energy level, so we have $\mathcal{N} = 4$.

In short we have shown that when $b > 0$ a global phase portrait of systems (III) is topologically equivalent to the phase portraits 1.5 and 1.6 of Figure 1 when $a = 0$ and $a > 0$, respectively. Consequently we obtain the bifurcation diagram shown in Figure 2.

3. Bifurcation diagram for systems (IV)

Systems (IV)

$$\dot{x} = ax + by - 3x^2y - y^3, \tag{3a}$$

$$\dot{y} = -\frac{a^2+1}{b}x - ay + 3xy^2, \tag{3b}$$

have the Hamiltonian

$$H_4(x, y) = -\frac{y^4}{4} - \frac{3}{2}x^2y^2 + \frac{a^2+1}{2b}x^2 + \frac{b}{2}y^2 + axy.$$

According to [7] a global phase portrait of systems (IV) is topologically equivalent to the phase portrait 1.1 of Figure 1 when $b < 0$. However there are four possibilities when $b > 0$, namely the phase portraits 1.2, 1.7, 1.8 and 1.9 of Figure 1. So we will only focus on the case $b > 0$. Note that besides the origin the phase portrait 1.2 has two finite singular points, 1.9 has four, and 1.7 and 1.8 both have six finite singular points. Moreover we observe that there are four saddles at the same energy level in the phase portrait, whereas an argument similar to the one used in Remark 3 proves that there are at most two finite saddles at a fixed energy level in the phase portrait 1.8. We are going to use these two basic properties to distinguish them.

We will first study the case $a = 0$ because it appears as a critical value in our calculations. In this case we can easily calculate the finite singular points of systems (IV) which are the origin, $(0, \pm\sqrt{b})$, and whenever $3b^2 > 1$ the additional four points

$$\left(\pm\frac{1}{3}\sqrt{\frac{3b^2-1}{b}}, \pm\sqrt{\frac{1}{3b}}\right). \tag{4}$$

Note that when $3b^2 - 1 = 0$ we get $1/3b = b$, and there are only two distinct singular points.

The linear part of systems (IV) with $a = 0$ is

$$M_4 = \begin{pmatrix} -6xy & b - 3x^2 - 3y^2 \\ -1/b + 3y^2 & 6xy \end{pmatrix},$$

The eigenvalues of M_4 at the singular points (4) are the same, so they are saddles because there are at most two centers or cusps. Furthermore, since

H_4 is even these saddles are at the same energy level. Thus a global phase portrait systems (IV) with $a = 0$ and $b > 0$ is topologically equivalent to the phase portrait 1.2 of Figure 1 if $b \leq 1/\sqrt{3}$, and to 1.7 if $b > 1/\sqrt{3}$.

We now assume $a > 0$ for the rest of this section. To find the number of finite singular points of systems (IV) we solve for x in the equation $\dot{x} = 0$ and get

$$x_{1,2} = \frac{a \pm \sqrt{a^2 + 12by^2 - 12y^4}}{6y}.$$

Note that when $y = 0$ (3a) becomes ax, so the only singular point on the x–axis is the origin. Since we are looking for singularities other than the origin we assume $y \neq 0$. We substitute x_1 and x_2 into (3b) and obtain $\dot{y}_1$ and $\dot{y}_2$, respectively:

$$\dot{y}_{1,2} = \frac{-a - a^3 - 3aby^2 \mp (1 + a^2 - 3by^2)\sqrt{a^2 + 12by^2 - 12y^4}}{6by}.$$

We claim that the number of distinct real roots N of the product $\dot{y}_1\dot{y}_2$

$$3y^6 - \frac{2 + 2a^2 + 3b^2}{b}y^4 + \frac{(1 + a^2)(1 + a^2 + 6b^2)}{3b^2}y^2 - \frac{1 + a^2}{3b} \tag{5}$$

is in one–to–one correspondence with the number of finite singular points M of systems (IV). We now prove our claim.

Let y_0 be a real root of (5). The corresponding x-coordinate x_0 is unique depending on whether y_0 is a root of $\dot{y}_1$ or $\dot{y}_2$, unless y_0 is a common root of $\dot{y}_1$ and $\dot{y}_2$ such that $a^2 + 12by^2 - 12y^4 \neq 0$. But if y_0 is a common root of $\dot{y}_1$ and $\dot{y}_2$, then we have

$$\dot{y}_1 + \dot{y}_2 = -\frac{a(1 + a^2 + 3b^2y_0^2)}{3by_0} \neq 0$$

for any $y_0 \in \mathbb{R}$ because $a \neq 0$. Therefore $\dot{y}_1$ and $\dot{y}_2$ cannot have a common root, and we have $M \leq N$.

On the other hand we have $M < N$ only if $a^2 + 12by_0^2 - 12y_0^4 < 0$ so that x_0 is complex. If we define

$$\begin{aligned} s_1 =& -a - a^3 - 3aby_0^2, \\ s_2 =& 1 + a^2 - 3by_0^2, \\ s_3 =& a^2 + 12by_0^2 - 12y_0^4, \end{aligned}$$

then y_0 is root of (5) if and only if $s_1^2 - s_3 s_2^2 = 0$. For x_0 to be complex we need $s_3 < 0$, which implies $s_1 = s_2 = 0$. But we see that $s_1 - as_2 = 2a(1+a^2) \neq 0$, which is a contradiction. Thus s_3 cannot be negative, and as a result we obtain $M = N$, proving the claim. We note that since systems (IV) have at least two finite singular points different from the origin, (5) must have at least two distinct real roots.

We will study the root classification of (5) using [24], where the author provides in particular the root classification of an arbitrary sextic polynomial of the form

$$x^6 + px^4 + qx^3 + rx^2 + sx^2 + t$$

We first need to compute the "discriminant sequence" $\{D_1, \dots, D_6\}$ where

$$
\begin{aligned}
D_1 &= 1, \quad D_2 = -p, \quad D_3 = 24rp - 8p^3 - 27q^2,\\
D_4 &= 32p^4r - 12p^3q^2 + 96p^3t + 324prq^2 - 224r^2p^2 - 288ptr - 120qp^2s\\
&\quad + 300ps^2 - 81q^4 + 324tq^2 - 720qsr + 384r^3,\\
D_5 &= -4p^3q^2r^2 - 1344ptr^3 + 24p^4q^2t + 144pq^2r^3 + 1440ps^2r^2 + 162q^4tp\\
&\quad - 5400rts^2 + 1512prtsq + 16p^4r^3 - 192p^4t^2 + 72p^5s^2 - 128r^4p^2\\
&\quad + 256r^5 + 1875s^4 - 64p^5rt + 592p^3tr^2 + 432rt^2p^2 - 616rs^2p^3\\
&\quad + 558q^2p^2s^2 + 1080s^2tp^2 - 2400ps^3q - 324pt^2q^2 - 1134tsq^3\\
&\quad + 648q^2tr^2 + 1620q^2s^2r - 1344qsr^3 + 3240qst^2 + 12p^3q^3s - 1296pt^3\\
&\quad - 27q^4r^2 + 81q^5s + 1728t^2r^2 - 56p^4rsq - 72p^3tsq + 432r^2p^2sq\\
&\quad - 648rq^2tp^2 - 486prq^3s,\\
D_6 &= -32400ps^2t^3 - 3750pqs^5 + 16q^3p^3s^3 - 8640q^2p^3t^3 + 825q^2p^2s^4\\
&\quad + 108q^4p^3t^2 + 16r^3p^4s^2 - 64r^4p^4t - 4352r^3p^3t^2 + 512r^2p^5t^2\\
&\quad + 9216rp^4t^3 - 900rp^3s^4 - 17280t^3p^2r^2 - 192t^2p^4s^2 + 1500tp^2s^4\\
&\quad - 128r^4p^2s^2 + 512r^5p^2t + 9216r^4pt^2 + 2000r^2s^4p + 108s^4p^5\\
&\quad - 1024p^6t^3 - 4q^2p^3r^2s^2 - 13824t^4p^3 + 16q^2p^3r^3t + 8208q^2p^2r^2t^2\\
&\quad - 72q^3p^3str + 5832q^3p^2st^2 + 24q^2p^4ts^2 - 576q^2p^4t^2r - 4536q^2p^2s^2tr\\
&\quad - 72rp^4qs^3 + 320r^2p^4qst - 5760rp^3qst^2 - 576rp^5ts^2 + 4816r^2p^3s^2t\\
&\quad - 120tp^3qs^3 + 46656t^3p^2qs - 6480t^2p^2s^2r + 560r^2qp^2s^3 - 2496r^3qp^2st\\
&\quad - 3456r^2qpst^2 - 10560r^3s^2pt + 768sp^5t^2q + 19800s^3rqpt + 3125s^6\\
&\quad - 46656t^5 - 13824r^3t^3 + 256r^5s^2 - 1024r^6t + 62208prt^4 + 108q^5s^3\\
&\quad - 874q^4t^3 + 729q^6t^2 + 34992q^2t^4 - 630prq^3s^3 + 3888prq^2t^3\\
&\quad + 2250rq^2s^4 - 4860prq^4t^2 - 22500rts^4 + 144pr^3q^2s^2 - 576pr^4q^2t\\
&\quad - 8640r^3q^2t^2 + 2808pr^2q^3st + 21384rq^3st^2 - 9720r^2q^2s^2t\\
&\quad - 77760rt^3qs + 43200r^2t^2s^2 - 1600r^3qs^3 + 6912r^4qst - 27540pq^2t^2s^2\\
&\quad - 27q^4r^2s^2 + 108q^4r^3t - 486q^5str + 162pq^4ts^2 - 1350q^3ts^3\\
&\quad + 27000s^3qt^2.
\end{aligned}
$$

Then we will determine the "sign list" $[\operatorname{sign}(D_1), \dots, \operatorname{sign}(D_6)]$ of the discriminant sequence, where the sign function is

$$
\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0,\\ 0 & \text{if } x = 0,\\ -1 & \text{if } x < 0. \end{cases}
$$

And finally we need to construct the associated "revised sign list" $[r_1, \dots, r_6]$ which will give all the information about the number of real and complex roots of our polynomial. Given any sign list $[s_1, \dots, s_n]$, the revised sign list $[r_1, \dots, r_n]$ is obtained as follows:

If $s_k \neq 0$ we write $r_k = s_k$.
If $[s_i, s_{i+1}, \dots, s_{i+j}]$ is a section of the given sign list such that $s_{i+1} = \dots = s_{i+j-1} = 0$ with $s_i s_{i+j} \neq 0$, then in place of $[r_{i+1}, \dots, r_{i+j-1}]$ we write the $(j-1)$-tuple

$$[-s_i, -s_i, s_i, s_i, -s_i, -s_i, s_i, s_i, -s_i, \dots].$$

Note that in this way there are no zeros between nonzero elements of the revised sign list.

The elements of the discriminant sequence of polynomial (5) are

$$D_2 = \frac{A}{3b}, \quad D_3 = \frac{8AB^2}{27b^3}, \quad D_4 = -\frac{32(1+a^2)B^2C}{243b^4},$$
$$D_5 = \frac{16a^2(1+a^2)^2CD}{6561b^6}, \quad D_6 = \frac{64a^4(1+a^2)^3D^2}{531441b^9},$$

where

$$\begin{aligned} A &= 2+2a^2+3b^2, \quad B = 1+a^2-3b^2, \\ C &= -2(1-3b^2)^2 + a^2(2+21b^2) + 4a^4, \\ D &= 4(1-3b^2)^3 + 3a^2(4+3b^2+36b^4) + 3a^4(4+15b^2) + 4a^6. \end{aligned} \tag{6}$$

We will determine the cases in which (5) has six or four distinct real roots, and consequently the case with two distinct real roots will follow.

It is given in [24] that (5) has six distinct real roots if and only if the revised sign list of its discriminant sequence is $[1, 1, 1, 1, 1, 1]$. Since $A > 0$, (5) has six distinct real roots if and only $B \neq 0$ and $C, D < 0$. We see that $D \leq 0$ only if $1 - 3b^2 < 0$, in which case

$$D - (a^2 - 2 + 6b^2)C = 9a^2(2 + 2a^2 + 2b^2) > 0.$$

This means that if $D \leq 0$ then $C < 0$ also. In addition when $B = 0$, that is $a = \sqrt{1-3b^2}$ and $1 - 3b^2 > 0$, we have $D > 0$. Consequently we have $B \neq 0$ if $D \leq 0$. Therefore we deduce that (5) has six distinct real roots if and only if $D < 0$.

We remind that systems (IV) have two global phase portraits with six finite singular points which are not topologically equivalent, namely the phase portraits 1.7 and 1.8 of Figure 1. We will prove below that systems (IV) with $D < 0$ and $a > 0$ cannot have four finite saddles at the same energy level, hence, as we mentioned in the beginning of this section, their phase portraits cannot be topologically equivalent to 1.7.

To determine the number of finite singular points at an energy level we look for the number of solutions of the system of three equations $\dot{x} = \dot{y} = 0$ and $H_4 = h$ for $h \in \mathbb{R}$. As we have shown for systems (III), we have $h > 0$ at finite singular points of systems (IV). We calculate the Gröebner basis of the polynomials $\dot{x}$, $\dot{y}$ and $H_4 - h$ and obtain 23 polynomials. The polynomials and the calculations are almost the same as those for systems (III). Among these 23 polynomials only three are enough for our study: one that is linear in the variable x with the coefficient $27a(1+a^2) > 0$, and two that do not contain the variable x and they are of the form $my^2 + n$. The coefficients of y^2 in these two polynomials are

$$6h(-1 + 90h - 1728h^2 + 5832b^2h^2 - 13824h^3),$$

$$3h(11 + 3a^2 + 18b^2 - 336h + 972b^2h - 2304h^2).$$

We know that $h > 0$. Then we need to check if the remaining non–constant factors can be zero simultaneously. The resultant of these two factors is

$$1253826625536D < 0.$$

Therefore at least one of these polynomials is not identically zero. Taking into account the third polynomial which is linear in x, we deduce that this system of equations have at most two solutions. As a result all the global phase portraits of systems (IV) when $D < 0$ and $a > 0$ are topologically equivalent to 1.8 of Figure 1.

Now we study when (5) has four distinct real roots. According to [24] the unique revised sing list of the discriminant sequence must be $[1, 1, 1, 1, 0, 0]$ because we have $D_6 \geq 0$. Hence we need $B \neq 0$, $C < 0$ and $D = 0$. We have already seen that $B \neq 0$ and $C < 0$ whenever $D = 0$. Therefore (5) has four distinct real roots if and only if $D = 0$.

As a result of the above analysis and the fact that (5) has at least two distinct real roots, it follows easily that (5) has two distinct real roots if and only if $D > 0$.

We observe that when $a = 0$ we have $D = 4(1 - 3b^2)^3$. Hence we can summarize our results as follows: When $b < 0$ then the global phase portraits of systems (IV) are topologically equivalent to 1.1 of Figure 1. When $b > 0$ the systems (IV) have the global phase portrait 1.2 of Figure 1 when $D > 0$ or $D = a = 0$, 1.7 if $D < 0$ and $a = 0$, 1.8 if $D < 0$ and $a > 0$, and finally 1.9 if $D = 0$ and $a > 0$. Therefore we obtain the bifurcation diagram shown in Figure 3.

4. Bifurcation diagram for systems (V)

Systems (V)

$$\dot{x} = ax + by - 3\mu x^2 y + y^3, \tag{7a}$$

$$\dot{y} = -\frac{a^2 + 1}{b}x - ay + x^3 + 3\mu x y^2, \tag{7b}$$

have the Hamiltonian

$$H_5(x, y) = \frac{y^4 - x^4}{4} - \frac{3\mu}{2}x^2y^2 + \frac{a^2 + 1}{2b}x^2 + \frac{b}{2}y^2 + axy.$$

Due to Theorem 1 the global phase portraits of systems (V) are topologically equivalent to the phase portraits 1.3, 1.10, 1.11 or 1.12 of Figure 1. Note that besides the origin the phase portrait 1.3 has two finite singular points, 1.12 has four, and 1.10 and 1.11 both have six finite singular points. Moreover there are four finite saddles at the same energy level in 1.10, while there can be at most two saddles at a fixed energy level in 1.11 (see Remark 3). We will use these facts to determine the bifurcation points for these phase portraits.

We first remark that without loss of generality we can assume $b > 0$. To prove this we apply the linear transformation $(t, x, y) \mapsto (-t, y, -x)$ to systems (V) and get

$$\begin{aligned} -\dot{y} &= ay - bx + 3\mu y^2 x - x^3, \\ \dot{x} &= -\frac{a^2+1}{b} y + ax + y^3 + 3\mu y x^2, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \dot{x} &= ax - \frac{a^2+1}{b} y + 3\mu x^2 y + y^3, \\ \dot{y} &= bx - ay + x^3 - 3\mu x y^2. \end{aligned} \tag{8}$$

After defining $\bar{\mu} = -\mu$, and $\bar{b} = -(a^2+1)/b$, we see that systems (8) are essentially systems (V) with $b\bar{b} < 0$. So we assume $b > 0$.

As we did for systems (IV) we study the case $a = 0$ separately. In this case the finite singular points besides the origin are $(\pm 1/\sqrt{b}, 0)$, and whenever $3\mu > b^2$ the four points

$$\left(\pm\sqrt{\frac{1+3b^2\mu}{b(1+9\mu^2)}}, \pm\sqrt{\frac{3\mu - b^2}{b(1+9\mu^2)}} \right). \tag{9}$$

Note that when $3\mu = b^2$ the four singular points in (9) coincide with $(\pm 1/\sqrt{b}, 0)$.

The linear part of systems (V) with $a = 0$ is

$$M_5 = \begin{pmatrix} -6\mu x y & b - 3\mu x^2 + 3y^2 \\ -1/b + 3x^2 + 3\mu y^2 & 6\mu x y \end{pmatrix}.$$

The eigenvalues of M_5 at the four singular points in (9) are the same. Since there are at most two centers or cusps, these singular points are saddles, and they are at the same energy level because H_5 is even. Therefore a global phase portrait of systems (V) with $a = 0$ is topologically equivalent to the phase portrait 1.3 of Figure 1 if $3\mu \le b^2$, and to 1.10 otherwise. This finishes the study of the case $a = 0$ and in the rest of this section we will assume that $a > 0$.

We start by determining the number of finite singular points of systems (V) as a function of the parameters a, b, μ. If we equate (7a) to zero, solve for x and substitute both roots into (7b) we obtain two functions of y. If multiply them we get a polynomial of degree eight instead of six, which was the case for systems (IV). Consequently it is more difficult to study the number of distinct real roots of this polynomial as a guide to determine the number of finite singular points of systems (V). Instead we use the fact that systems (V) are symmetric with respect to the origin, and look for pairs of finite singular points different from the origin which lie on straight lines passing through the origin. Therefore we study systems (V) on the y–axis, and on the lines $y = cx$ for $c \in \mathbb{R} \setminus \{0\}$. We can assume $c \neq 0$ due to the fact that when $c = 0$ we have $y = 0$, and (7a) becomes ax, which means that the only singular point is the origin. We will identify the lines $y = cx$ by the parameter c.

On the y–axis (7b) becomes $-ay$, which means that the only singular point is the origin. So we assume $x \neq 0$ and impose $y = cx$ to rewrite systems (V) as

$$\dot{x} = (a+bc)x + c(c^2-3\mu)x^3, \tag{10a}$$

$$\dot{y} = -\frac{1+a^2+abc}{b}x + (1+3\mu c^2)x^3. \tag{10b}$$

We equate (10a) to zero, solve for x and get

$$x = \pm\sqrt{\frac{-a-bc}{c(c^2-3\mu)}}. \tag{11}$$

We see that (11) are not defined if $\mu > 0$ and $c = \pm\sqrt{3\mu}$. So we will now find out if there are singular points on these lines that are different from the origin.

If $c = \sqrt{3\mu}$ then (10a) becomes $(a + b\sqrt{3\mu})x \neq 0$ because $a, b, \mu > 0$ and $x \neq 0$. Thus the only singular point on this line is the origin. If $c = -\sqrt{3\mu}$ then (10a) becomes $(a - b\sqrt{3\mu})x$, which is zero if and only if $a = b\sqrt{3\mu}$, in which case equating (10b) to zero and solving for x gives

$$x = \pm\frac{1}{\sqrt{b(1+9\mu^2)}} = \pm\frac{b^{3/2}}{\sqrt{a^4+b^4}}, \tag{12}$$

which are real and nonzero. Therefore when $c = -\sqrt{3\mu}$ there are singular points other than the origin if and only if $a = b\sqrt{3\mu}$. We will keep this in mind and continue looking for singular points with $c \neq -\sqrt{3\mu}$.

We substitute (11) into (10b) and obtain

$$\pm\frac{\sqrt{-a-bc}(abc^4 + (1+a^2+3b^2\mu)c^3 + (b^2-3(1+a^2)\mu)c + ab)}{b(c(c^2-3\mu))^{3/2}}.$$

This means that at a singular point we must have

$$P_5(c) = abc^4 + (1+a^2+3b^2\mu)c^3 + (b^2-3(1+a^2)\mu)c + ab = 0$$

because $-a-bc = 0$ yields $x = 0$. Moreover in order that x defined in (12) are real and nonzero, the roots of P must satisfy

$$Q_5(c) = (-a-bc)c(c^2-3\mu) > 0$$

so that (11) are real and nonzero. Then each real root of P_5 will yield a pair of finite singular points different from the origin. Here the index 5 is a reminder that we are studying systems (V).

We will study the number of distinct real roots of P_5 using [24], where the elements of the discriminant sequence of an arbitrary quartic polynomial

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$$

are given as

$$\begin{aligned}
D_1 &= 1, \quad D_2 = 3a_1^2 - 8a_2a_0,\\
D_3 &= 16a_0^2a_4a_2 - 18a_0^2a_3^2 - 4a_0a_2^3 + 14a_0a_3a_1a_2 - 6a_0a_4a_1^2 + a_2^2a_1^2\\
&\quad - 3a_3a_1^3,\\
D_4 &= 256a_0^3a_4^3 - 27a_0^2a_3^4 - 192a_0^2a_3a_4^2a_1 - 27a_1^4a_4^2 - 6a_0a_1^2a_4a_3^2\\
&\quad + a_2^2a_3^2a_1^2 - 4a_0a_2^3a_3^2 + 18a_2a_4a_1^3a_3 + 144a_0a_2a_4^2a_1^2\\
&\quad - 80a_0a_2^2a_4a_1a_3 + 18a_0a_2a_3^3a_1 - 4a_2^3a_4a_1^2 - 4a_1^3a_3^3 + 16a_0a_2^4a_4\\
&\quad - 128a_0^2a_2^2a_4^2 + 144a_0^2a_2a_4a_3^2.
\end{aligned} \tag{13}$$

By using them we will be able to determine the exact number of distinct real roots of P_5.

Note that the number of real roots of Q_5 are different when $\mu \leq 0$ and $\mu > 0$. We will investigate these separately.

Case $\mu \leq 0$. In this case we have $Q_5 > 0$ if and only if $c \in (-a/b, 0)$, see Figure 10. On the other hand we have $P_5(0) = ab > 0$ and $P_5(-a/b) = a(3b^2\mu - a^2)/b^3 < 0$, so P_5 has at least one root in $(-a/b, 0)$. In fact we observe that P_5 has either two or zero negative roots due to Descartes' rule of sign. Additionally it has at least one root in $(-\infty, -a/b)$ because $\lim_{c\to-\infty} P_5 = \infty$. Therefore when $\mu \leq 0$ P_5 has exactly one real root in $(-a/b, 0)$, and systems (V) have only two finite singular points other than the origin.

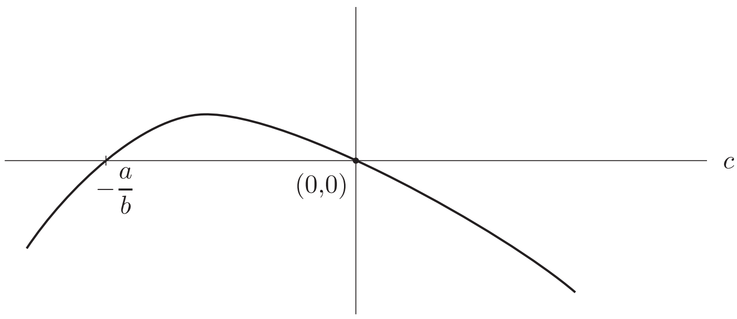

FIGURE 10. A rough graph of $Q_5(c)$ when $\mu \leq 0$.

Case $\mu > 0$. Now Q_5 has the four roots $c = 0$, $c = -a/b$ and $c = \pm\sqrt{3\mu}$. Moreover

$$\begin{aligned}
P_5(-a/b) &= \frac{a(3b^2\mu - a^2)}{b^3}, \quad P_5(0) = ab,\\
P_5(\pm\sqrt{3\mu}) &= b(a \pm b\sqrt{3\mu})(1 + 9\mu^2),
\end{aligned} \tag{14}$$

at these points. Since the roots $c = -a/b$ and $c = -\sqrt{3\mu}$ are independent of each other we will investigate this case in three subcases comparing a/b to $\sqrt{3\mu}$.

When $a/b < \sqrt{3\mu}$ we have $Q_5 > 0$ if and only if $c \in (-\sqrt{3\mu}, -a/b) \cup (0, \sqrt{3\mu})$, see Figure 11. Thus we will look for the number of real roots P_5 in these intervals.

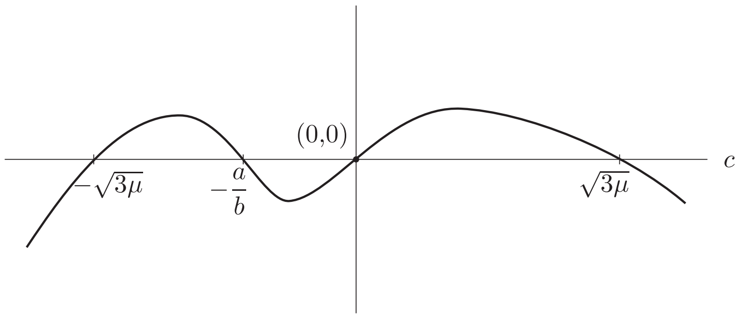

FIGURE 11. A rough graph of $Q_5(c)$ when $\mu > 0$ and $a/b < \sqrt{3\mu}$.

We have $P_5(-\sqrt{3\mu}) < 0$ and $P_5(-a/b) > 0$, see (14). Since P_5 has at most two negative roots and $\lim_{c\to-\infty} P_5 = \infty$, P_5 has exactly one simple root in $(\sqrt{3\mu}, -a/b)$.

On the other hand we have $P_5(0) > 0$ and $P_5(\sqrt{3\mu}) > 0$. We claim that P_5 cannot have a real root greater that $\sqrt{3\mu}$. This is due to the fact that the first derivative of P_5 with respect to c,

$$P_5'(c) = 4abc^3 + 3(1 + a^2 + 3b^2\mu)c^2 + b^2 - 3(1 + a^2)\mu, \tag{15}$$

has at most one positive root. If P_5 had a real root greater than $\sqrt{3\mu}$ then it would have at least two positive critical points because $P_5(\sqrt{3\mu}) > P_5(0)$ and $\lim_{c\to\infty} P_5 = \infty$. Therefore if P_5 has a positive root then it is in $(0, \sqrt{3\mu})$.

In short when $a/b < \sqrt{3\mu}$, P_5 has at least two real simple roots (the negative ones), and exactly one of its real roots (the smallest) makes (11) complex.

When $a/b > \sqrt{3\mu}$ everything is the same as in the case $a/b < \sqrt{3\mu}$, except that the roles of the roots $c = -a/b$ and $c = -\sqrt{3\mu}$ are exchanged, see Figure 12. More precisely we have $Q_5 > 0$ if and only if $c \in (-a/b, -\sqrt{3\mu}) \cup (0, \sqrt{3\mu})$. In addition $P_5(-a/b) < 0$ and $P_5(-\sqrt{3\mu}) > 0$ so that P_5 has one negative root in $(-a/b, -\sqrt{3\mu})$, and a smaller one in $(-\infty, -a/b)$. Moreover $P_5(\sqrt{3\mu}) > P_5(0) > 0$, and hence any positive root of P_5 is in the interval $(0, \sqrt{3\mu})$ because it has at most one positive critical point, see (15). Therefore P_5 has at least two real simple roots, and exactly one of them leads to a pair of complex singular points when $a/b > \sqrt{3\mu}$.

Finally when $a/b = \sqrt{3\mu}$ we see that $Q_5 > 0$ only when $c \in (0, a/b)$, see Figure 13. Hence no negative root of P_5 satisfies $Q_5 > 0$. We recall that when $a/b = \sqrt{3\mu}$ there *are* extra singular points on the line $y = cx$ with $c = -\sqrt{3\mu} = -a/b$. Also $P_5(-a/b) = 0$, and thus $c = -a/b$ *is* a root of P_5. Moreover $P_5'(-\sqrt{3\mu}) = 6\mu + b^2(1 + 9\mu^2) > 0$ so that it is a simple root. Then since P_5 has either two or zero negative roots, it has another negative root different from $-a/b$. By the same argument used in the previous two subcases, all the positive roots of P_5 are in $(0, \sqrt{3\mu})$, and

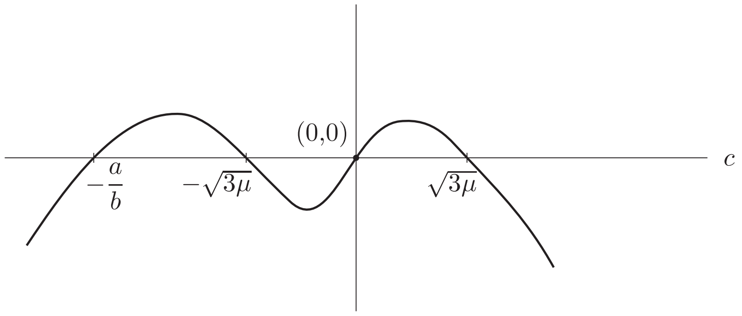

FIGURE 12. A rough graph of $Q_5(c)$ when $\mu > 0$ and $a/b > \sqrt{3\mu}$.

they satisfy $Q_5 > 0$. Therefore P_5 again has at least two simple roots with the property that exactly one of them correspond to complex finite singular points.

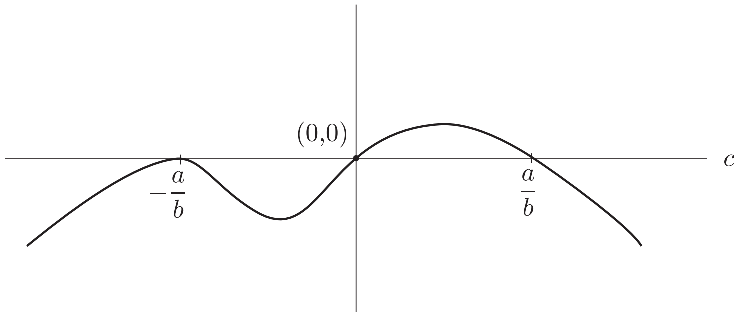

FIGURE 13. A rough graph of $Q_5(c)$ when $\mu > 0$ and $a/b = \sqrt{3\mu}$.

In short we have shown that in any case P_5 has at least two simple real roots when $\mu > 0$. Then according to [24] P_5 has two, three or four distinct real roots if and only if $D_4 < 0$, $D_4 = 0$ and $D_4 > 0$, respectively, where

$$\begin{aligned}
D_4 = & - b^2(27a^2 + 108a^4 + 162a^6 + 108a^8 + 27a^{10} + 4b^4 + 18a^2b^4 \\
& + 216a^4b^4 - 54a^6b^4 + 27a^2b^8) + 36b^4(3a^2-1)^2\big((1+a^2)^2 \\
& - b^4\big)\mu - 54b^2(2 + 11a^2 + 24a^4 + 26a^6 + 14a^8 + 3a^{10} - 6b^4 \\
& + 32a^2b^4 + 50a^4b^4 + 12a^6b^4 + 2b^8 + 3a^2b^8)\mu^2 + 108\big((1+a^2)^2 \\
& - b^4\big)(1 + 4a^2 + 6a^4 + 4a^6 + a^8 - 8b^4 + 8a^2b^4 + 16a^4b^4 + b^8)\mu^3 \\
& - 243b^2(-4 - 11a^2 - 4a^4 + 14a^6 + 16a^8 + 5a^{10} + 12b^4 + 38a^2b^4 \\
& + 40a^4b^4 + 14a^6b^4 - 4b^8 + 5a^2b^8)\mu^4 + 2916b^4\big((1+a^2)^2 - b^4\big) \\
& (1+a^2)^2\mu^5 + 2916b^6(1+a^2)^3\mu^6,
\end{aligned} \tag{16}$$

see (13). Since there are no additional finite singular points at exactly one simple root of P_5, systems (V) have two, four and six additional finite singular points besides the origin whenever $D_4 < 0$, $D_4 = 0$ and $D_4 > 0$, respectively.

Note that the phase portraits 1.10 and 1.11 have the same number of singular points. We observe that there are four finite singular points at a fixed energy level in 1.10. So we will check if the Hamiltonian H_5 can attain the same value at four distinct finite singular points.

At a singular point of systems (V) H_5 reduces to

$$H_5(x,y) - \frac{y\dot{x} - x\dot{y}}{4} = \frac{x^2 + (ax+by)^2}{4b}. \tag{17}$$

If we substitute $y = cx$ in (17) we get

$$G_5(c,x) = \frac{\big(1 + (a+bc)^2\big)x^2}{4b}.$$

Then using (11) we can rewrite G_5 as

$$F_5(c) = -\frac{(a+bc)\big(1+(a+bc)^2\big)}{4bc(c^2 - 3\mu)}.$$

We recall that there are additional singular points on the line $c = -\sqrt{3\mu}$ if and only if $a/b = \sqrt{3\mu}$, and that (11) is not well defined at $c = -\sqrt{3\mu}$. Thus to calculate H_5 at the additional singular points on the line $c = -\sqrt{3\mu}$ we must use G_5, while we can use F_5 for all the other singular points. Therefore H_5 can attain the same value at four singular points only if one of the following holds:

(i) If F_5 attains the same value at two distinct real roots of P_5 which are different from $-\sqrt{3\mu}$.

(ii) If $a/b = \sqrt{3\mu}$ and $F_5(c) = G_5(-a/b, \pm b^{3/2}/\sqrt{a^4+b^4})$ for a real root $c \neq -a/b$ of P_5 (see (12)).

We claim that none of these two cases holds. The proof is as follows.

To show that (i) cannot hold we assume on the contrary that c_1 and c_2 are two distinct real roots of P_5 which satisfy

$$F_5(c_1) - F_5(c_2) = \frac{(c_1 - c_2)E_5(c_1,c_2)}{4bc_1c_2(c_1^2 - 3\mu)(c_2^2 - 3\mu)} = 0,$$

where

$$\begin{aligned} E_5(c_1,c_2) = {}& a(1+a^2)(c_2^2 - 3\mu) + c_1c_2\big(a(1+a^2+9b^2\mu) + bc_2(1+3a^2 \\ & + 3b^2\mu)\big) + c_1^2\big(a(1+a^2) + bc_2(1+3a^2+3b^2\mu+3abc_2)\big) \end{aligned}$$

Since $c_1 \neq c_2$ we have $F_5(c_1) = F_5(c_2)$ if and only if $E_5(c_1,c_2) = 0$. To find a necessary condition on the parameters a, b, μ so that we have $P_5(c_1) = P_5(c_2) = E_5(c_1,c_2) = 0$ we compute the resultant $R(c_2)$ of $P_5(c_1)$ and $E_5(c_1,c_2)$ with respect to c_1, and then compute the resultant of $R(c_2)$ and $P_5(c_2)$ with respect to c_2. Doing so we obtain

$$-a^{10}b^{10}(1+2a^2+a^4+b^4)^2(a^2 - 3b^2\mu)^7(1+9\mu^2)^2D_4^3, \tag{18}$$

which is equal to zero only if $a^2 = 3b^2\mu$ because we have $a, b > 0$ and $D_4 > 0$ when systems (V) have six finite singular points. But when $a^2 = 3b^2\mu$ we have $E_5(c_1,c_2) = 0$ only if either $c_1 = -\sqrt{3\mu} = -a/b$ or $c_2 = -a/b$, which cannot be in case (i). More precisely if we set $\mu = a^2/3b^2$ we get

$$P_5(c) = \frac{(a+bc)\bar{P}_5(c)}{b^2}, \quad E_5(c_1,c_2) = -\frac{(a+bc_1)(a+bc_2)\bar{E}_5(c_1,c_2)}{b^2},$$

where

$$\bar{P}_5(c) = -b^3 + a(1+a^2)c - b(1+a^2)c^2 - ab^2c^3,$$
$$\bar{E}_5(c_1, c_2) = -a(1+a^2) + b(1+a^2)(c_1+c_2) + 3ab^2c_1c_2.$$

If we impose $c_1, c_2 \neq -a/b$, then $P_5(c_1) = P_5(c_2) = E_5(c_1, c_2) = 0$ if and only if $\bar{P}_5(c_1) = \bar{P}_5(c_2) = \bar{E}_5(c_1, c_2) = 0$. But if we calculate the resultant $\bar{R}(c_2)$ of $\bar{P}_5(c_1)$ and $\bar{E}_5(c_1, c_2)$ with respect to c_1, and then the resultant of $\bar{R}(c_2)$ and $\bar{P}_5(c_2)$ with respect to c_2 we get

$$-\frac{a^3b^{18}D_4}{(2a^2+a^4+b^4)^2} \neq 0.$$

Therefore (18) cannot be zero, and hence (i) cannot hold.

On the other hand if (ii) holds then substituting $\mu = a^2/3b^2$ gives

$$F_5(c) = \frac{b\big(1+(a+bc)^2\big)}{4c(a-bc)}, \quad G_5\left(-\frac{a}{b}, \pm\frac{b^{3/2}}{\sqrt{a^4+b^4}}\right) = \frac{b^2}{4(a^4+b^4)} = h.$$

Since a root c of P_5 which is different from $-a/b$ must satisfy $F_5(c) = h$, we must have $\bar{P}_5(c) = 0$. However, the resultant of $\bar{P}_5$ and $F_5 - h$ with respect to c is

$$-\frac{b^2(2a^2+a^4+b^4)^3(1+2a^2+a^4+b^4)}{64(a^4+b^4)^3} \neq 0.$$

This disproves (ii). Hence when $a, \mu > 0$ at most two singular points can be at the same energy level, and systems (V) cannot have the phase portrait 1.10 of Figure 1. This completes the case $\mu > 0$.

We note that when $a = 0$ we have $D_4 = 4(1+3b^2\mu)^3(3\mu - b^2)^3$, so the sign of D_4 is enough to determine the phase portraits. Therefore we can summarize our results as follows: when $\mu \leq 0$ a global phase portrait of systems (V) is topologically equivalent to 1.3 of Figure 1; when $\mu > 0$ then it is equivalent to 1.3 if $D_4 < 0$ or $D_4 = 0$ and $a = 0$, to 1.10 if $D_4 > 0$ and $a = 0$, to 1.11 if $D_4 > 0$ and $a \neq 0$, and to 1.12 if $D_4 = 0$ and $a \neq 0$. Hence when $\mu \leq 0$ there is a unique phase portrait, and when $\mu > 0$ we obtain the bifurcation diagram shown in Figure 4.

5. Bifurcation diagram for systems (VI)

Systems (VI)

$$\dot{x} = ax + by - 3\mu x^2 y - y^3, \quad \dot{y} = -\frac{a^2+1}{b}x - ay + x^3 + 3\mu xy^2.$$

have the Hamiltonian

$$H_6(x,y) = -\frac{y^4+x^4}{4} - \frac{3\mu}{2}x^2y^2 + \frac{a^2+1}{2b}x^2 + \frac{b}{2}y^2 + axy.$$

Due to [7] a global phase portrait of systems (VI) is topologically equivalent to the phase portraits 1.13 and 1.14 of Figure 1 if $\mu < -1/3$, to 1.15–1.17 if $\mu = -1/3$, and to 1.18–1.23 if $\mu > -1/3$. Therefore we will determine the bifurcation points of the global phase portraits of systems (VI) when $\mu < -1/3$, $\mu = -1/3$ and $\mu > -1/3$ separately.

Case $\mu < -1/3$. First of all we note that without loss of generality we can assume in this case that $b > 0$. Indeed, if we rotate the coordinate axes by $\pi/4$ via the linear transformation $(x, y) \mapsto \big((x-y)/\sqrt{2}, (x+y)/\sqrt{2}\big) = (u, v)$ then systems (VI) become

$$\begin{aligned}\dot u &= \frac{a^2-b^2+1}{2b}u + \frac{(a+b)^2+1}{2b}v - \frac{3(1-\mu)}{2}u^2v - \frac{1+3\mu}{2}v^3,\\ \dot v &= -\frac{(a-b)^2+1}{2b}u - \frac{a^2-b^2+1}{2b}v + \frac{3(1-\mu)}{2}uv^2 + \frac{1+3\mu}{2}u^3.\end{aligned}$$

If we further rescale the independent variable by $d\tau = (1+3\mu)/2\ dt$ then we get

$$\begin{aligned}\dot u &= \frac{a^2-b^2+1}{b(1+3\mu)}u + \frac{(a+b)^2+1}{b(1+3\mu)}v - \frac{3(1-\mu)}{1+3\mu}u^2v - v^3,\\ \dot v &= -\frac{(a-b)^2+1}{b(1+3\mu)}u - \frac{a^2-b^2+1}{b(1+3\mu)}v + \frac{3(1-\mu)}{1+3\mu}uv^2 + u^3.\end{aligned}$$

An finally after defining the parameters

$$\bar a = \frac{a^2-b^2+1}{b(1+3\mu)},\quad \bar b = \frac{(a+b)^2+1}{b(1+3\mu)},\quad \bar\mu = \frac{1-\mu}{1+3\mu},$$

we get the systems

$$\begin{aligned}\dot u &= \bar a u + \bar b v - 3\bar\mu u^2 v - v^3,\\ \dot v &= -\frac{\bar a^2+1}{\bar b}u - \bar a v + 3\bar\mu uv^2 + u^3.\end{aligned} \tag{19}$$

We see that $d\bar\mu/d\mu = -4/(1+3\mu)^2 < 0$ and $\lim_{\mu\to-\infty}\bar\mu = -1/3$, hence $\bar\mu < -1/3$ whenever $\mu < -1/3$. As a result systems (19) are basically systems (VI) with $b \mapsto -b$, proving that we can assume $b > 0$.

We remind that according to [7] systems (VI) can have two different global phase portraits when $\mu < -1/3$, namely 1.13 and 1.14 of Figure 1. Both phase portraits have the same number of singular points. But the difference between them is that there are four finite singular points at the same energy level in 1.14, whereas there are only two in 1.13 because otherwise using the same arguments used in Remark 3 we can find a straight line through the origin that intersects the separatrices of the saddles six times. So we will investigate when there can be four finite singular points at a fixed energy level.

When $a = 0$ the finite singular points of systems (VI) besides the origin are $(\pm 1/\sqrt{b}, 0)$ and $(0, \pm\sqrt{b})$. We also have

$$H_6\left(\pm\frac{1}{\sqrt{b}}, 0\right) = \frac{1}{4b^2},\quad H_6\left(0, \pm\sqrt{b}\right) = \frac{b^2}{4}. \tag{20}$$

Hence the four singular points are on the same energy level if and only if $b = 1$. Therefore a global phase portrait of systems (VI) with $\mu < -1/3$ and $a = 0$ is topologically equivalent to 1.13 of Figure 1 if $b \neq 1$, and to 1.14 if $b = 1$.

We now assume $a > 0$ and consider finite singular points of systems (VI) which are different from the origin in pairs lying on the straight lines $y = cx$ with $c \in \mathbb{R} \setminus \{0\}$. We note that there are no finite singular points on the coordinate axes because $a > 0$. We will again identify each line $y = cx$ with its parameter c.

We substitute $y = cx$ in systems (VI) and we get

$$\dot{x} = (a + bc)x - c(c^2 + 3\mu)x^3, \tag{21a}$$

$$\dot{y} = -\frac{1 + a^2 + abc}{b}x + (1 + 3\mu c^2)x^3. \tag{21b}$$

Then we equate (21a) to zero and solve for x to obtain

$$x = \pm\sqrt{\frac{a + bc}{c(c^2 + 3\mu)}}. \tag{22}$$

We see that (22) are not defined if $c = \pm\sqrt{-3\mu}$. However, when $c = \sqrt{-3\mu}$ we get from (21a) that $\dot{x} = (a + b\sqrt{-3\mu})x \neq 0$, and there are no additional singular points on this line. When $c = -\sqrt{-3\mu}$ we have $\dot{x} = (a - b\sqrt{-3\mu})x$, which is zero if and only if $a = b\sqrt{-3\mu}$. But if we substitute $c = -\sqrt{-3\mu}$ and $a = b\sqrt{-3\mu}$ in (21b) then the roots of $\dot{y}$ become

$$x = \pm 1/\sqrt{b(1 - 9\mu^2)} \tag{23}$$

which are complex because $\mu < -1/3$. Therefore (22) are well–defined at the singular points.

We proceed as we did for systems (V). We can substitute (22) into (21b) to get

$$\pm\frac{\sqrt{a + bc}\big(abc^4 + (1 + a^2 - 3b^2\mu) + (3(1 + a^2)\mu - b^2)c - ab\big)}{b\big(c(c^2 + 3\mu)\big)^{3/2}}.$$

Therefore at a singular point different from the origin we must have

$$P_6(c) = abc^4 + (1 + a^2 - 3b^2\mu) + (3(1 + a^2)\mu - b^2)c - ab = 0 \tag{24}$$

and

$$Q_6(c) = (a + bc)c(c^2 + 3\mu) > 0. \tag{25}$$

Now that we know the necessary and sufficient conditions for a point to be a finite singular point of systems (VI) with $\mu < -1/3$, we check if four singular points can be at the same energy level. We follow the same way we used for systems (V).

At a singular point H_6 can be written exactly as (17). Then we substitute $y = cx$ and obtain

$$G_6(c, x) = \frac{\big(1 + (a + bc)^2\big)x^2}{4b}, \tag{26}$$

If we further substitute (22) in G_6 we get

$$F_6(c) = \frac{(a + bc)\big(1 + (a + bc)^2\big)}{4bc(c^2 + 3\mu)}. \tag{27}$$

Thus systems (VI) with $\mu < -1/3$ have four finite singular points at the same energy level if and only if P_6 has two distinct real roots c_1 and c_2 such that $F_6(c_1) = F_6(c_2)$ and $Q_6(c_{1,2}) > 0$. We now prove that this is possible if and only if $b = \sqrt{1 + a^2}$.

Assume that $P_6(c_1)=P_6(c_2)=0$ but $c_1\neq c_2$. We have

$$F_6(c_1)-F_6(c_2)=\frac{(c_1-c_2)E_6(c_1,c_2)}{4bc_1c_2(c_1^2+3\mu)(c_2^2+3\mu)}=0,$$

if and only if $E_6(c_1,c_2)=0$ where

$$\begin{aligned}E_6(c_1,c_2)=\;&a(1+a^2)(c_2^2+3\mu)+c_1c_2\big(a(1+a^2-9b^2\mu)\\&+bc_2(1+3a^2-3b^2\mu)\big)+c_1^2\big(a(1+a^2)\\&+bc_2(1+3a^2-3b^2\mu+3abc_2^2)\big).\end{aligned}\tag{28}$$

If we calculate the resultant $R(c_2)$ of $P_6(c_1)$ and $E_6(c_1,c_2)$ with respect to c_1, and then the resultant of $P_6(c_2)$ and $R(c_2)$ we obtain

$$a^{10}b^{10}(1-9\mu^2)^2(a^2+3b^2\mu)^7\big(b^4-(1+a^2)^2\big)^2D_4^3,\tag{29}$$

where

$$\begin{aligned}D_4=&-b^2(27a^2+108a^4+162a^6+108a^8+27a^{10}-4b^4\\&-18a^2b^4-216a^4b^4+54a^6b^4+27a^2b^8)-36b^4(3a^2-1)^2\\&\big((1+a^2)^2+b^4\big)\mu+54b^2(2+11a^2+24a^4+26a^6+14a^8\\&+3a^{10}+6b^4-32a^2b^4-50a^4b^4-12a^6b^4+2b^8+a^2b^8)\mu^2\\&-108\big((1+a^2)^2+b^4\big)(1+4a^2+6a^4+4a^6+a^8+8b^4\\&-8a^2b^4-16a^4b^4+b^8)\mu^3-242b^2(-4-11a^2-4a^4+14a^6\\&+16a^8+5a^{10}-12b^4-38a^2b^4-40a^4b^4-14a^6b^4-4b^8\\&+5a^2b^8)\mu^4-2916b^4(1+a^2)^2\big((1+a^2)^2+b^4\big)\mu^5\\&+2916b^6(1+a^2)^3\mu^6.\end{aligned}\tag{30}$$

We remark that we denote (30) by D_4 because it coincides with the fourth element of the discriminant sequence of $P_6(c)$, see (13). Since $a,b>0$ we see that (29) is zero only if (i) $D_4=0$, (ii) $\mu=-a^2/(3b^2)$, or (iii) $b=\sqrt{1+a^2}$. We now analyze these three cases.

We first prove that (i) cannot hold, that is $D_4\neq 0$. We observe that D_4 is equal to the "standard" discriminant of $P_6(c)$. Hence we can prove that (i) cannot hold by showing that all the roots of P_6 are simple. The roots of Q_6 are 0, $-a/b$ and $\pm\sqrt{-3\mu}$. If we evaluate P_6 at these points we get

$$\begin{aligned}&P_6(0)=-ab,\quad P_6(-a/b)=-a(a^2+3b^2\mu)/b^3,\\&P_6(\pm\sqrt{-3\mu})=b(a\pm b\sqrt{-3\mu})(9\mu^2-1).\end{aligned}\tag{31}$$

If $a/b<\sqrt{-3\mu}$ then Q_6 is positive if and only if $c\in(-\infty,-\sqrt{-3\mu})\cup(-a/b,0)\cup(\sqrt{-3\mu},\infty)$, see Figure 14. Since $a,b>0$ we have $P_6(-\sqrt{-3\mu})<0$, $P_6(-a/b)>0$, $P_6(0)<0$ and $P_6(\sqrt{-3\mu})>0$, see (31). Since P_6 is a quartic polynomial in c it has four simple roots, two of which satisfy $Q_6>0$ as expected because the global phase portraits have four finite singular points.

If $a/b>\sqrt{-3\mu}$ then we have $Q_6(c)>0$ if and only if $c\in(-\infty,-a/b)\cup(-\sqrt{-3\mu},0)\cup(\sqrt{-3\mu},\infty)$, see Figure 15. In this case we have $P_6(-a/b)<0$, $P_6(-\sqrt{-3\mu})>0$, $P_6(0)<0$ and $P_6(\sqrt{-3\mu})>0$. Hence P_6 has four simple roots again, and exactly two of its roots are in the region where $Q_6>0$.

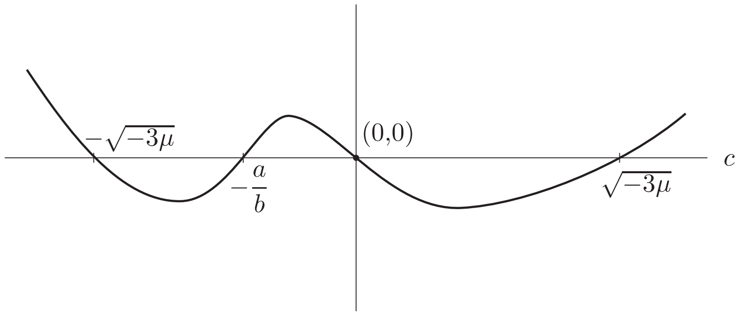

FIGURE 14. A rough graph of $Q_6(c)$ when $\mu < 0$ and $a/b < \sqrt{-3\mu}$.

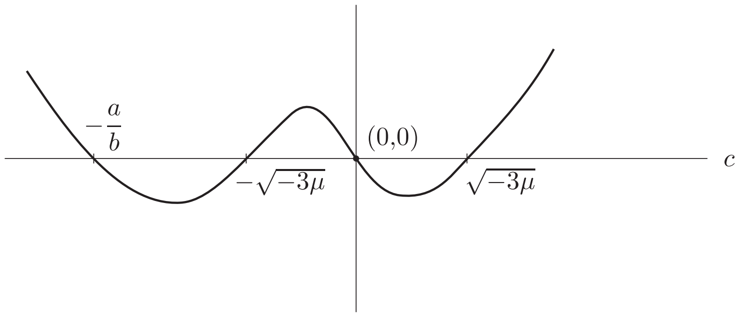

FIGURE 15. A rough graph of $Q_6(c)$ when $\mu < 0$ and $a/b > \sqrt{-3\mu}$.

Finally if $a/b = \sqrt{-3\mu}$, we have $Q_6(c) > 0$ unless $c \in \{-a/b\} \cup [0, \sqrt{-3\mu}]$, see Figure 16. We see that P_6 has at least one positive root with $Q_6 < 0$. In addition $P_6(-a/b) = 0$, and thus at least two distinct roots of P_6 do not satisfy $Q_6 > 0$. But since we know that P_6 has exactly two distinct roots with $Q_6 > 0$, we conclude that each root of P_6 is simple.

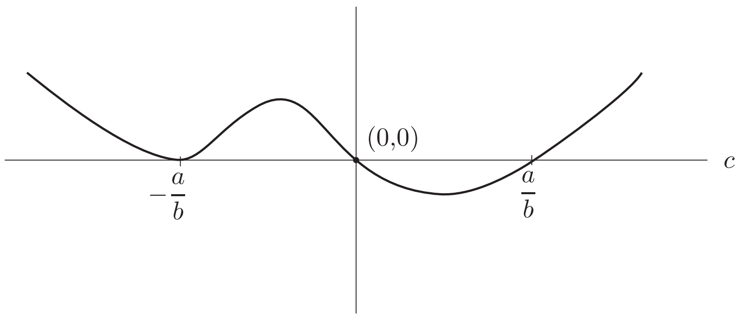

FIGURE 16. A rough graph of $Q_6(c)$ when $\mu < 0$ and $a/b = \sqrt{-3\mu}$.

In any case P_6 has four simple real roots, therefore $D_4 \neq 0$ and (i) does not hold.

We now consider (ii). Note that the conditions $P_6(c_1) = P_6(c_2) = 0$ and $E_6(c_1, c_2) = 0$ do not imply $Q_6(c_{1,2}) > 0$. We claim that when $\mu = -a^2/(3b^2)$

the resultant (29) vanishes only if $Q_6(c_1)Q_6(c_2) = 0$. We now prove this claim by showing that if we do not allow $Q_6(c_1)Q_6(c_2) = 0$ then (29) cannot be zero. The proof is as follows.

If we substitute $\mu = -a^2/(3b^2)$ in P_6 and E_6 we obtain

$$P_6(c) = -\frac{(a+bc)\bar{P}_6(c)}{b^2}, \; E_6(c_1,c_2) = -\frac{(a+bc_1)(a+bc_2)\bar{E}_6(c_1,c_2)}{b^2}, \tag{32}$$

where

$$\begin{aligned} \bar{P}_6(c) &= b^3 + a(1+a^2)c_2 - b(1+a^2)c_2^2 - ab^2c^3, \\ \bar{E}_6(c_1,c_2) &= -a(1+a^2) + b(1+a^2)(c_1+c_2) + 3ab^2c_1c_2. \end{aligned} \tag{33}$$

Since we have $Q_6(-a/b) = 0$ (see Figure 16), systems (VI) have four finite singular points at the same energy level if and only if $\bar{P}_6(c_1) = \bar{P}_6(c_2) = \bar{E}_6(c_1,c_2) = 0$. But if, as we did above, calculate first the resultant $\bar{R}(c_2)$ of $\bar{P}_6(c_1)$ and $\bar{E}_6(c_1,c_2)$ with respect to c_1, and then the resultant of $\bar{P}_6(c_2)$ and $\bar{R}(c_2)$ with respect to c_1 we get

$$\begin{aligned} -a^3b^{12}(a^2 + 8a^4 + 18a^6 + 16a^8 + 5a^{10} + 4b^4 + 30a^2b^4 + 48a^4b^4 \\ +22a^6b^4 - 27a^2b^8)^3, \end{aligned} \tag{34}$$

which is different from zero because when $\mu = -a^2/(3b^2)$ we have

$$\begin{aligned} D_4 =& \frac{(2a^2+a^4-b^4)^2}{b^6}(a^2 + 8a^4 + 18a^6 + 16a^8 + 5a^{10} + 4b^4 + 30a^2b^4 \\ &+ 48a^4b^4 + 22a^6b^4 - 27a^2b^8) \neq 0 \end{aligned} \tag{35}$$

This finishes the proof of our claim, and hence there cannot be four finite singular points at the same energy level when (ii) holds.

Finally we consider (iii). We remark that in this case we have $a^2/b^2 = 1 - 1/b^2 < 1$ whereas $-3\mu > 1$, and thus cases (ii) and (iii) are disjoint. If we substitute $b = \sqrt{1+a^2}$ in (24) we get

$$P_6(c) = \sqrt{1+a^2}(c^2-1)\left(a + \sqrt{1+a^2}(1-3\mu)c + ac^2\right). \tag{36}$$

Since we have $a/b < 1 < \sqrt{-3\mu}$, the roots $c = \pm 1$ of (36) make $Q_6 < 0$, see Figure 14. In addition we know that P_6 has two distinct roots with $Q_6 > 0$, so they must be the remaining two roots, which are

$$c_{1,2} = \frac{\sqrt{1+a^2}(3\mu-1) \pm \sqrt{(1+a^2)(1-3\mu)^2 - 4a^2}}{2a}. \tag{37}$$

Indeed, substituting (37) into (28) gives $E_6 = 0$. Therefore we conclude that systems (VI) with $\mu < -1/3$ have four singular points at the same energy level if and only if $1 + a^2 = b^2$.

We observe that when $a = 0$ the condition $1 + a^2 = b^2$ translates into $b = 1$, which is the unique case in which systems (VI) have four finite singular points at a fixed energy level. Consequently we have proved that systems (VI) with $\mu < -1/3$ have the global phase portrait 1.14 of Figure 1 if $b = \sqrt{1+a^2}$, and the phase portrait 1.13 otherwise. Therefore we obtain the bifurcation diagram shown in Figure 5.

Case $\mu = -1/3$. In [7] it is shown that if $b < 0$ the unique phase portrait is 1.15 of Figure 1. So we study the case $b > 0$, in which a global phase portrait of systems (VI) is topologically equivalent to either 1.16 or 1.7 of Figure 1. As in the case $\mu < -1/3$, these two phase portraits differ in the sense that in 1.17 there exists an energy level at which there are four finite singular points, whereas 1.16 has at most two finite singular points at a fixed energy level. This follows from applying the argument used in Remark 3. Therefore we will study the number of finite singular points at a fixed energy level of systems (VI) with $\mu = -1/3$.

When $a = 0$ the finite singular points of systems are the same as in the case $\mu < -1/3$ and they are at the same energy level if and only if $b = 1$, see (20). Thus a global phase portrait is topologically equivalent to 1.17 of Figure 1 if $b = 1$, and to 1.16 otherwise.

Assume now that $a > 0$. Following the same way we used in the case $\mu < -1/3$, we rewrite systems (VI) by substituting $\mu = -1/3$ in (21) and we get

$$\dot{x} = (a + bc)x - c(c^2 - 1)x^3, \tag{38a}$$

$$\dot{y} = -\frac{1 + a^2 + abc}{b}x - (c^2 - 1)x^3. \tag{38b}$$

In addition (22) becomes

$$x = \pm\sqrt{\frac{a + bc}{c(c^2 - 1)}}. \tag{39}$$

It is easy to check that on the lines $c = \pm 1$ the only singular point is the origin. Since we are looking for singular points other than the origin, we suppose $c \neq \pm 1$ and substitute (39) into (38b) to obtain

$$\mp\frac{\sqrt{a + bc}(c^2 - 1)\big(ab + (1 + a^2 + b^2)c + abc^2\big)}{b\big(c(c^2 - 1)\big)^{3/2}}. \tag{40}$$

Since we want $x \neq 0$, the roots of (40) we are interested in are

$$c_{1,2} = -\frac{1 + a^2 + b^2 \pm \sqrt{(1 + a^2 + b^2)^2 - 4a^2b^2}}{2ab}. \tag{41}$$

We notice that c_1 and c_2 in (41) are real and distinct. Due to the fact that systems (38) have at least four finite singular points, (41) must make (39) real and nonzero. Now we will check if these four singular points can be at the same energy level.

When $\mu = -1/3$ we see that (27) becomes

$$F_6(c) = \frac{(a + bc)\big(1 + (a + bc)^2)\big)}{4bc(c^2 - 1)}$$

Then we have

$$F_6(c_1) - F_6(c_2) = -\frac{(1 + a^2 - b^2)\sqrt{(1 + a^2 + b^2)^2 - 4ab^2}}{4b^2},$$

which is zero if and only if $b = \sqrt{1 + a^2}$. We remind that when $a = 0$ the four singular points are at the same energy level if and only if $b = 1$, which coincides with $b = \sqrt{1 + a^2}$.

In short we have the following result: When $b < 0$ systems (VI) with $\mu = -1/3$ have the global phase portrait 1.15 of Figure 1, and when $b > 0$ their global phase portraits are topologically equivalent to 1.17 if $b = \sqrt{1+a^2}$, and to 1.16 otherwise. Thus the bifurcation diagram shown in Figure 6 is obtained.

Case $\mu > -1/3$. Due to [7] a global phase portrait of systems (VI) in this case is topologically equivalent to one of the phase portraits 1.18–1.23 of Figure 1. Due to the direction of the flow at infinity the unique global phase portrait when $b < 0$ is clearly 1.18, so we only need to study systems (VI) with $b > 0$. It is also shown in [7] that the global phase portrait 1.23 is obtained if and only if $a = 0$, $b = 1$ and $\mu = 1/3$. Hence we will focus on the phase portraits 1.19–1.22.

In order to distinguish these phase portraits, we will use the properties that allowed us to distinguish the phase portraits of the previous families of systems. More precisely, the phase portrait 1.19 has four finite singular points, 1.22 has six, and 1.20 and 1.21 both have eight finite singular points besides the origin. Moreover 1.20 has four finite saddles at some fixed energy energy level, whereas 1.21 has at most two. This is again due to the same argument used in Remark 3.

As we did for the previous systems we will study the cases $a = 0$ and $a > 0$ separately. We note that this time the case $a = 0$ is a little more complicated so we will further divide the case $\mu > -1/3$ into the two corresponding subcases.

Subcase $a = 0$. The finite singular points of systems (VI) other than the origin are $(\pm 1/\sqrt{b}, 0)$ and $(0, \pm\sqrt{b})$, and the additional four points

$$\left(\sqrt{\frac{1-3b^2\mu}{b(1-9\mu^2)}}, \sqrt{\frac{b^2-3\mu}{b(1-9\mu^2)}}\right) \tag{42}$$

if $\mu < 1/3$, $1 - 3b^2\mu > 0$ and $b^2 - 3\mu > 0$, or if $\mu > 1/3$, $1 - 3b^2\mu < 0$ and $b^2 - 3\mu < 0$. Note that when $(1 - 3b^2\mu)(b^2 - 3\mu) = 0$ the singular points in (42) coincide with $(\pm 1/\sqrt{b}, 0)$ or $(0, \pm\sqrt{b})$. We also point out that if $1 - 3b^2\mu = b^2 - 3\mu = 0$ then we have $b = 1$ and $\mu = 1/3$, and thus there are infinitely many singular points (see the phase portrait 1.23).

We see that when $a = 0$ systems (VI) with $\mu > -1/3$ have either four or eight finite singular points besides the origin. When it has four finite singular points, clearly their phase portraits are topologically equivalent to 1.19. When it has eight finite singular points, there are two possibilities, namely the phase portraits 1.20 and 1.21. We now analyze these two possible cases.

The eigenvalues of the linear part of systems (VI) with $a = 0$ at each of the singular points (42) are

$$\pm\sqrt{\frac{4(1-3b^2\mu)(b^2-3\mu)}{b^2(9\mu^2-1)}}.$$

Hence they are centers if $\mu < 1/3$, and saddles if $\mu > 1/3$. Consequently the remaining finite singular points $(\pm 1/\sqrt{b}, 0)$ and $(0, \pm\sqrt{b})$ are saddles if

$\mu < 1/3$, and centers if $\mu > 1/3$. Now we shall check if the Hamiltonian H_6 can attain the same value at all the saddles.

When $\mu < 1/3$ we have $H_6(\pm 1/\sqrt{b}, 0) = 1/4b^2$ and $H_6(0, \pm\sqrt{b}) = b^2/4$ at the saddles. Therefore we have the phase portrait 1.20 if and only if $b = 1$, and 1.21 otherwise.

When $\mu > 1/3$, at each of the the singular points (42) the Hamiltonian becomes

$$\frac{1 + b^4 - 6b^2\mu}{4b^2(1 - 9\mu^2)},$$

hence we only have the phase portrait 1.20.

In short when $a = 0$ we obtain the bifurcation diagram shown in (7).

Subcase $a > 0$. The calculations in this case are very similar to the case $\mu < -1/3$, so we will often refer to the ones in the case $\mu < -1/3$.

Since $a > 0$ there are no additional finite singular points on the $y - axis$, so we substitute $y = cx$ with $c \neq 0$ as usual and obtain systems (21). We solve for x by equating (21a) to zero and obtain (22). This time, however, we see that there are additional singular points on the line $c = -\sqrt{-3\mu}$ if and only if $a = b\sqrt{-3\mu}$ because (23) are real when $-1/3$. Therefore similar to what we did for systems (V) we will keep this in mind and look for singular points with $c \neq \sqrt{-3\mu}$. Then (22) are well-defined, and we can substitute them in (21b) to see that we must have $P_6 = 0$ (see (24)) and $Q_6 > 0$ (see (25)) at the finite singular points.

The roots of Q_6 in this case are 0, $-a/b$, and additionally $\pm\sqrt{-3\mu}$ whenever $\mu < 0$. At these points P_6 becomes as in (31). Moreover the graph of Q_6 is roughly the one shown in Figure 17 if $\mu \geq 0$, the one in Figure 14 if $\mu < 0$ and $a < b\sqrt{-3\mu}$, the one in Figure 15 if $\mu < 0$ and $a > b\sqrt{-3\mu}$, and the one in Figure 16 if $\mu < 0$ and $a = b\sqrt{-3\mu}$. We will show now that any root of P_6 that is different from $c = -\sqrt{-3\mu}$ satisfy $Q_6 > 0$. We recall that $c = -\sqrt{-3\mu}$ is a root only when $a = b\sqrt{-3\mu}$. As a result we will conclude that the number of additional finite singular points of systems (VI) in this case is equal to the number of real distinct roots of P_6.

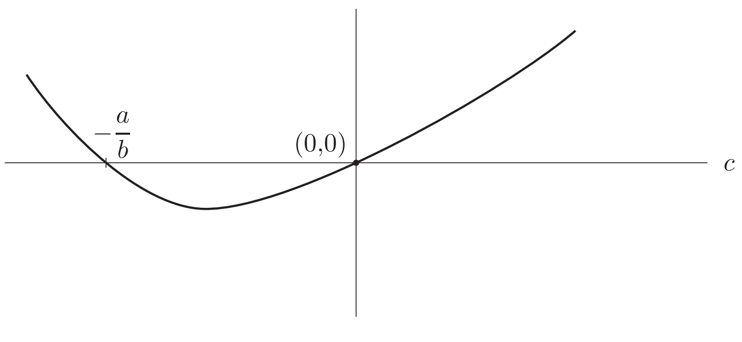

FIGURE 17. A rough graph of $Q_6(c)$ when $\mu \geq 0$.

We see that $P_6(\sqrt{-3\mu}) < 0$, and that P_6 has exactly one positive real root. Since $\lim_{c\to\infty} P_6 = \infty$, this positive root is greater than $\sqrt{-3\mu}$, and hence satisfies $Q_6 > 0$. Therefore it remains to show that P_6 does not have a negative root

(i) in $(-a/b, 0)$ when $\mu \geq 0$,
(ii) in $(-\sqrt{-3\mu}, -a/b)$ when $\mu < 0$ and $a < b\sqrt{-3\mu}$,
(iii) in $(-a/b, -\sqrt{-3\mu})$ when $\mu < 0$ and $a > b\sqrt{-3\mu}$.

To determine if a polynomial has a root in some interval we will use the following lemma.

Lemma 4. *If a polynomial $p(x)$ of degree n has a real root in an interval (α, β), then the polynomial $(1+x)^n(p \circ h)(x)$ with $h(x) = (\alpha + \beta x)/(1+x)$ has a positive root.*

Proof. Clearly $(1+x)^n(p \circ h)(x)$ is a polynomial. We have $h(0) = a$ and $\lim_{x\to\infty} h(x) = b$. In addition $h'(x) = (b-a)/(1+x^2) > 0$ so that h is bijective. Therefore if $p(x) = 0$ for some point $x_0 \in (\alpha, \beta)$ then $p(h(x_1)) = 0$ for $x_1 = h^{-1}(x_0) > 0$. □

To study roots of type (i) we define $h(c) = (-a/b)/(1+c)$. Since the degree of P_6 is four we have

$$\begin{aligned}(1+c)^4 P_6(h(c)) = -\frac{a}{b^3}\big(&a^2 + 3b^2\mu + (a^2 + a^4 + b^4 + 9b^2\mu + 6a^2b^2\mu)c \\ &+ 3b^2(b^2 + 3\mu + 3a^2\mu)c^2 + 3b^2(b^2 + \mu + a^2\mu)c^3 + b^4c^4\big),\end{aligned}$$

which does not have a positive root due to Descartes' rule of signs. Hence P_6 does not have a root of type (i).

We remark that in Lemma 4 although we chose $h(x) = (\alpha + \beta x)/(1+x)$, we could as well choose $h(x) = (\beta + \alpha x)/(1+x)$. Thus P_6 has a root of type (ii) if and only if it has a root of type (iii), so we only study (ii). To simplify notation will write $m = \sqrt{-3\mu}$. Hence we have $0 < m < 1$, $a > bm$ and

$$(1+c)^4 P_6(h(c)) = -\frac{a - bm}{b^3} S(c),$$

where

$$\begin{aligned}S(c) =& b^4(1 - m^4) + b^2(3b^2 + 2m^2 + 2a^2m^2 - 4abm^3 - b^2m^4)c \\ &+ 3b(b^3 + am + a^3m + bm^2 - a^2bm^2 - ab^2m^3)c^2 \\ &+ (a^2 + a^4 + b^4 + 4abm + b^2m^2 - 2a^2b^2m^2)c^3 \\ &+ a(a + bm)c^4\end{aligned}$$

We claim that S does not have a positive root, and we now prove this claim by showing that the sign of the coefficients of the monomials in S are all positive. The constant term and the coefficient of c^4 are clearly positive, so we look at the coefficients of c, c^2 and c^3.

The coefficient of c^3, which we denote by k_3, is

$$\begin{aligned}a^2 + a^4 + b^4 + 4abm + b^2m^2 - 2a^2b^2m^2 &> a^2 + a^4 + b^4 - 2a^2b^2 \\ &= a^2 + (a^2 - b^2)^2 > 0\end{aligned}$$

because $0 < m < 1$.

The coefficient of c^2, denoted by k_2, has exactly one positive root due to Descartes' rule of signs when considered as a polynomial in m. Moreover that root is greater than 1 due to the facts that when $m = 1$ we have

$$\begin{aligned} k_2 &= 3b(b^3 + a + a^3 + b - a^2b - ab^2) \\ &= 3b\big((a+b)(a^2 - ab + b^2) + a + b - ab(a+b)\big) \\ &= 3b(a+b)\big(1 + (a-b)^2\big) > 0, \end{aligned}$$

and that it is negative for large $m > 0$. Therefore k_2 is positive for $0 < m < 1$.

Finally the coefficient k_1 of c is also positive because of the same reasons: it has a exactly one positive root when considered as a polynomial in m; it is negative for large $m > 0$; and when $m = 1$ it becomes $2\big(1+(a-b)^2\big) > 0$. This proves that S does not have a positive root, which in turn implies that P_6 has no root of types (ii) or (iii).

In short the number of finite singular points other than the origin is double the number of distinct real roots of P_6. Since we know that the phase portraits 1.19–1.22 have at least four finite singular points additional to the origin, P_6 must have at least two real distinct roots. Therefore, according to [24], a global phase portrait of systems (VI) in this case is topologically equivalent to 1.19 of Figure 1 if $D_4 < 0$ or $D_4 = D_3 = 0$, to 1.22 if $D_4 = 0$ and $D_3 \neq 0$, and to 1.20 or 1.21 if $D_4 > 0$, where D_4 is given in (30) and

$$\begin{aligned} D_3 = {} & 3b^2(+1 + 5a^2 + 7a^4 + 3a^6 - 6a^2b^4) - 9(1+a^2)(1 + 3a^2 + 3a^4 \\ & + a^6 + 3b^4 - 5a^2b^4)\mu + 27b^2(3 + 3a^2 - 3a^4 - 3a^6 + 3b^4 \\ & + 5a^2b^4)\mu^2 - 81b^4(3 + 6a^2 + 3a^4 + b^4)\mu^3 + 243(1+a^2)b^6\mu^4, \end{aligned} \tag{43}$$

see (13).

Note that when $D_4 > 0$ there are two phase portraits. We know that there are four saddles at a fixed energy level in 1.20, but there are at most two in 1.21. Hence following the exact same steps that we used in distinguishing the phase portraits 1.13 and 1.14 of systems (VI) with $\mu < -1/3$, we deduce that the phase portrait 1.20 is achieved only if (29) is zero. So we should investigate whether (29) can be zero.

We have $D_4 \neq 0$. When $\mu = 1/3$ we have

$$P_6(c) = (1 + c^2)(-ab + c + a^2c - b^2c + abc^2),$$

and hence the phase portrait is topologically equivalent to 1.19. So it remains to study the cases $\mu = -a^2/(3b^2)$ and $b = \sqrt{1+a^2}$.

When $\mu = -a^2/(3b^2)$, we have $-1/3 < \mu < 0$. Moreover, due to the results obtained in the case $\mu < -1/3$, if there are four finite singular points at the same energy level then two of these singular points must be on the line $c = -a/b$ (see (32), (33) and (34)). The Hamiltonian H_6 at the singular points when $c = -a/b$ is given by (26). The x–coordinates of the singular points when $c = -a/b$ are (23), and at these singular points the Hamiltonian becomes

$$G_6(c) = \frac{b^2}{4(b^4 - a^4)} = h.$$

Note that $0 < a^2/b^2 = -3\mu < 1$ so that $b^4 - a^4 > 0$. Now we should check if there are other singular points at which $H_6 = h$. For the singular points that are not on the line $c = -a/b$ we have $H_6 = F_6$ (see (27)), and these singular points satisfy $\bar{P}_6 = 0$ (see (33)). Hence we are looking for points that satisfy $\bar{P}_6 = F_6 - h = 0$. If we calculate the resultant of $\bar{P}_6$ and $F_6 - h$ we obtain

$$-\frac{b^2\big(b^4 - (1+a^2)^2\big)(2a^2 + a^4 - b^4)}{(64(b^4 - a^4)^3)},$$

which is zero if and only if $b = \sqrt{1+a^2}$, see (35). Note that this is the last condition that makes (29) zero, so now we will study this final case.

If we substitute $b = \sqrt{1+a^2}$ in (24) we get (36). We have seen that the roots (37) are at the same energy level. What remains to be done is to determine when these points are saddles and when they are centers. For reasons of simplicity we study the local phase portraits of the singular point on the lines $c = \pm 1$. If we evaluate (28) at $(1, -1)$ we obtain

$$-4a(1+a^2)(1+3\mu) \neq 0 \tag{44}$$

because $a > 0$. So the singular points on these lines cannot be at the same energy level. Hence we will deduce that if these singular points are centers then (37) are saddles and we have the phase portrait 1.20 of Figure 1, and if they are saddles then we have 1.21.

The linear part of systems (21) when $b = \sqrt{1+a^2}$ is

$$\begin{pmatrix} a - 6\mu c x^2 & \sqrt{1+a^2} - 3c^2x^2 - 3\mu x^2 \\ -\sqrt{1+a^2} + 3x^2 + 3\mu c^2 x^2 & -a + 6\mu c x^2 \end{pmatrix}. \tag{45}$$

When $c = \pm 1$ the x–coordinates of the singular points are obtained by (22). Then we see that the determinant of (45) is

$$d_1 = \frac{4\big(1 - 3a(a + \sqrt{1+a^2})(1-\mu) - 3\mu)\big)}{1+3\mu}$$

when $c = 1$, and it is

$$d_2 = \frac{4\big(1 - 3a(a - \sqrt{1+a^2})(1-\mu) - 3\mu)\big)}{1+3\mu}$$

when $c = -1$. If we multiply them we get

$$d_1 d_2 = \frac{16\big((1-3\mu)^2 - 3a^2(1-\mu)(1+3\mu)\big)}{(1+3\mu)^2}.$$

We observe that if we substitute $b = \sqrt{1+a^2}$ in D_4 we obtain

$$4(1+a^2)^3\big((1-3\mu)^2 - 3a^2(1-\mu)(1+3\mu)\big)^3. \tag{46}$$

Since we assume $D_4 > 0$ we have $d_1 d_2 > 0$, so meaning that they are different from zero and they have the same sign. Due to the fact that the eigenvalues of the linear part M of a Hamiltonian system is of the form $\pm\sqrt{-\det(M)}$, where $\det(M)$ denotes the determinant of M, we deduce that the singular points that are on the lines $c = \pm 1$ are saddles if $d_1 < 0$, and are centers if $d_1 > 0$.

Since d_1 is linear in μ we can solve $d_1 = 0$ and get

$$\mu_0 = 1/3 + 2a/(3\sqrt{1+a^2}).$$

So we have $d_1 > 0$ and $d_1 < 0$ for $\mu < \mu_0$ and $\mu > \mu_0$, respectively. On the other hand if we equate (46) to zero and solve for μ we get

$$\mu_{1,2} = 1/3 \mp 2a/(3\sqrt{1+a^2}),$$

Note that $\mu_0 = \mu_2$ and $-1/3 < \mu_1 < \mu_0$. Hence (46) is positive if and only if $\mu < \mu_1$ or $\mu > \mu_0$. Thus whenever $D_4 > 0$ we have $d_1 > 0$ if $\mu < \mu_1$, and $d_1 < 0$ if $\mu > \mu_0$. Therefore we get the global phase portrait 1.20 if $\mu < \mu_1$, and 1.21 if $\mu > \mu_0$. This finishes the analysis of the subcase $a > 0$.

Before summarizing our results for the case $\mu > -1/3$, we comment on the relation between the subcases $a = 0$ and $a > 0$. When $a = 0$ we have $\mu_0 = \mu_1 = 1/3$, and the condition $b = \sqrt{1+a^2}$ becomes $b = 1$. So for $\mu < \mu_1$ the conditions to have the phase portrait 1.20 when $a = 0$ can be obtained by substituting $a = 0$ in those when $a > 0$. However, for $\mu > \mu_0$, (44) becomes zero if $a = 0$, meaning that the saddles are at the same energy level as well as the centers, and we get the phase portrait 1.20 again. On the other hand, when $a = 0$ we have

$$\begin{aligned} D_4 &= 4(b^2 - 3\mu)^3(1 - 3b^2\mu)^3, \\ D_3 &= 3(b^2 - 3\mu)(1 - 3b^2\mu)^3. \end{aligned}$$

So the conditions for the phase portrait 1.19 when $a = 0$ can also be obtained by substituting $a = 0$ in those when $a > 0$. And finally the conditions for the phase portrait 1.22 when $a > 0$ can also be extended to $a = 0$ due the fact that when $a = 0$ we do not have $D_4 = 0$ and $D_3 \neq 0$, and also we do not have the phase portrait 1.22.

In short we obtain that when $b < 0$ a global phase portrait is topologically equivalent to 1.18 of Figure 1. When $b > 0$ a global phase portrait is topologically equivalent to 1.19 if $D_4 < 0$, or $D_4 = D_3 = 0$ but either $a \neq 0$, $\mu \neq 1/3$ or $b \neq 1$; to 1.20 if $D_4 > 0$, $b = \sqrt{1+a^2}$ and $\mu < \mu_1$, or $D_4 > 0$, $a = 0$ and $\mu > 1/3$; to 1.21 if $D_4 > 0$ and $b \neq \sqrt{1+a^2}$, or $D_4 > 0$, $b = \sqrt{1+a^2}$, $a \neq 0$ and $\mu > \mu_0$; to 1.22 if $D_4 = 0$ but $D_3 \neq 0$; and to 1.23 if $a = 0$, $\mu = 1/3$ and $b = 1$. Therefore we obtain the bifurcation diagrams shown in Figures 5–9.

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Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
E-mail address: ilkercolak@mat.uab.cat

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
E-mail address: jllibre@mat.uab.cat

Departamento de Matemática, Instituto Superior Técnico, Universidade Técnica de Lisboa, 1049–001 Lisboa, Portugal
E-mail address: cvalls@math.ist.utl.pt